

# GLOBAL EXISTENCE OF NULL-FORM WAVE EQUATIONS ON SMALL ASYPTOTICALLY EUCLIDEAN MANIFOLDS

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**ABSTRACT.** We prove the global existence of the small solutions to the Cauchy problem for quasilinear wave equations satisfying the null condition on  $(\mathbb{R}^3, \mathbf{g})$ , where the metric  $\mathbf{g}$  is a small perturbation of the flat metric and approaches the Euclidean metric like  $(1 + |x|^2)^{-\rho/2}$  with  $\rho > 1$ . Global and almost global existence for systems without the null condition are also discussed for certain small time-dependent perturbations of the flat metric in the appendix.

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## 1. INTRODUCTION

This paper is concerned with the Cauchy problem for the system of the quasi-linear wave equations in three space dimensions of the form

$$(1.1) \quad \begin{cases} \partial_t^2 u^I - c_I^2 \Delta_{\mathbf{g}} u^I = Q_{JK}^{I,\alpha\beta\gamma} \partial_\alpha u^J \partial_\beta \partial_\gamma u^K + S_{JK}^{I,\alpha\beta} \partial_\alpha u^J \partial_\beta u^K, I = 1, \dots, M \\ u^I(0, x) = u_0^I, \partial_t u^I(0, x) = u_1^I \end{cases}$$

subject to suitably small initial conditions, posed on certain asymptotically Euclidean manifolds  $(\mathbb{R}^3, \mathbf{g})$ . We assume that the propagation speeds,  $c_I > 0$ , are distinct, and we refer to this situation as the nonrelativistic case.

We shall construct a unique global classical solution, provided that the coefficients of the nonlinear terms satisfy the null condition and the metric  $\mathbf{g}$  is a small perturbation of the flat Euclidean metric and approaches the Euclidean metric like  $(1 + |x|^2)^{-\rho/2}$  with  $\rho > 1$ .

In the Minkowski space-time (with flat metric  $\mathbf{g}$ ), this problem has been extensively studied. In the relativistic case (with same  $c_I$ ), the null condition was first identified by Klainerman and shown to have global existence of small solutions in Christodoulou [6] and Klainerman [29] (see also Hörmander [11] and John [16]). Without the null condition, we have almost global existence [18], [28], [16], but as examples show, arbitrarily small initial conditions can blow up in finite time for some equations [15], [44]. Notice that this does not mean that the null condition is the necessary condition for the general quadratic quasilinear problems to have global existence with small data. Actually, there is a larger class of nonlinearities which will ensure global existence, which is related to the so-called “weak null condition” of Lindblad and Rodnianski, see e.g. [35], [36], [4], [24].

Small solutions always have global existence in higher dimensions [31], [43], [28]. The two-dimensional relativistic case is rather more complicated. The sharp results are given in [2], [3], with previous works in [12], [19].

In the non-relativistic case, the null condition guarantees that the self-interaction of each wave family is nonresonant and is the natural one for systems of quasilinear wave equations with multiple speeds. It is equivalent to the requirement that no plane wave solution of the system is genuinely nonlinear, which follows from the so-called “John-Shatah’s observation” (John [17] (p. 23) and [1]). Related results were established, e.g., by Kovalyov [34], Agemi and Yokoyama [1], Hoshiga and Kubo [14], Yokoyama [52], Sideris and Tu [46], Kubota and Yokoyama [33], Sogge [47], [48], Hidano [8], Katayama [20] [21], Katayama and Yokoyama [25], and Katayama and Kubo [22]. It is remarkable that the approach of [46] and [8] does not use estimates of the fundamental solution for the free wave equation, which is

more robust when considering problems with variable coefficients. The two-dimensional case has also been examined, see [13], [14], [8] and references therein.

In exterior domains, null form quasilinear wave equations were previously studied by Keel, Smith and Sogge [26], Metcalfe and Sogge [40] [42], Metcalfe, Nakamura and Sogge [38] [39], and Katayama and Kubo [23]. The general quasilinear problems were also studied, see [27], [41] and references therein.

It is interesting to investigate similar problems on various space-time manifolds. Recently, there have been some progress in this direction. Global and almost global existence of the semilinear problems posed on asymptotically Euclidean manifolds have been obtained in Bony and Häfner [5], Sogge and the first author [49], and the authors [50]. Global existence for the null form semilinear wave equations on slowly rotating Kerr spacetimes or time dependent inhomogeneous media has been obtained in Luk [37] and Yang [51]. In this paper, we will deal with the nonrelativistic quasilinear wave equations on small asymptotically Euclidean manifolds, mainly inspired by the approach of [46] and [8], together with the recent local energy estimates with variable coefficients obtained in Hidano, the first author and Yokoyama [9]. It will also be very interesting if we can deal with the problems with general non-trapping asymptotically Euclidean manifolds or small asymptotically flat manifolds with certain time-dependent metrics.

Before stating our main result, we introduce the necessary notations. Points in  $\mathbb{R}^4$  will be denoted by  $(x^0, x^1, x^2, x^3) = (t, x)$ . Partial derivatives will be written as  $\partial_\alpha = \partial/\partial x^\alpha$ , with the abbreviations  $\partial = (\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla)$ . Here, we have used the convention that Greek indices range from 0 to 3 and Latin indices from 1 to 3. We will also abuse the notation to use the Greek indices to denote multi-indices, which should be clear in the context. Hereafter, the Einstein summation convention will be performed over repeated indices. The angular-momentum operators are defined as

$$\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla,$$

where  $\wedge$  denotes the usual vector cross product in  $\mathbb{R}^3$ , and the scaling operator is defined by

$$(1.2) \quad S = t\partial_t + r\partial_r = x^\alpha \partial_\alpha, \quad r = |x|.$$

The collection of these eight vector fields will be labeled as

$$\Gamma = (\Gamma_0, \dots, \Gamma_7) = (\partial, \Omega, S).$$

We consider the asymptotically Euclidean Riemannian manifolds  $(\mathbb{R}^3, \mathbf{g})$  with

$$\mathbf{g} = g_{ij}(x) dx^i dx^j.$$

The metric  $\mathbf{g}$  is assumed to be a small perturbation of the flat metric. More precisely, we suppose  $g_{ij}(x) \in C^\infty(\mathbb{R}^3)$  and, for some fixed  $\rho > 0$  and  $\delta \ll 1$ ,

$$(H1) \quad \forall \alpha \in \mathbb{N}^3 \quad |\partial_x^\alpha (g_{ij} - \delta_{ij})| \leq C_\alpha \delta \langle x \rangle^{-|\alpha|-\rho},$$

with  $\delta_{ij} = \delta^{ij}$  being the Kronecker delta function and  $\langle x \rangle = \sqrt{1 + |x|^2}$ . Since  $\delta \ll 1$ , it is clear that the metric  $\mathbf{g}$  is a non-trapping perturbation. Let  $g = \det(g_{ij})$ , the Laplace–Beltrami operator associated with  $\mathbf{g}$  is given by

$$\Delta_{\mathbf{g}} = \sqrt{g}^{-1} \partial_i g^{ij} \sqrt{g} \partial_j,$$

where  $(g^{ij}(x))$  denotes the inverse matrix of  $(g_{ij}(x))$ .

Consider the initial value problem for the nonlinear equations of the form

$$(1.3) \quad (\square_{\mathbf{g}} u)^I \equiv (\partial_t^2 - c_I^2 \Delta_{\mathbf{g}}) u^I = N^I(u, u), I = 1, 2, \dots, M$$

in which the quadratic nonlinearity  $N = Q + S$  is of the form

$$(1.4) \quad Q^I(u, v) = Q_{JK}^{I, \alpha\beta\gamma} \partial_{\alpha} u^J \partial_{\beta} \partial_{\gamma} v^K, S^I(u, v) = S_{JK}^{I, \alpha\beta} \partial_{\alpha} u^J \partial_{\beta} v^K.$$

The construction of solutions will depend on the energy integral method, which requires the quasilinear part to be symmetric:

$$(1.5) \quad Q_{JK}^{I, \alpha\beta\gamma} = Q_{JK}^{I, \alpha\gamma\beta} = Q_{JI}^{K, \alpha\beta\gamma}.$$

The key assumption for global existence is the following *null* condition which says that the self-interaction of each wave family is nonresonant:

$$(1.6) \quad Q_{II}^{I, \alpha\beta\gamma} \xi_{\alpha} \xi_{\beta} \xi_{\gamma} = S_{II}^{I, \alpha\beta} \xi_{\alpha} \xi_{\beta} = 0 \quad \text{for all } \xi \text{ s.t. } \xi_0^2 = c_I^2(\xi_1^2 + \xi_2^2 + \xi_3^2).$$

The standard energy norm is denoted as

$$E_1(u(t)) = \frac{1}{2} \sum_{I=1}^M \int_{\mathbb{R}^3} |\partial u^I(t, x)|^2 dx,$$

and higher order derivatives will be estimated through

$$(1.7) \quad E_m(u(t)) = \sum_{|\alpha| \leq m-1} E_1(\Gamma^{\alpha} u(t)), \quad m = 2, 3, \dots$$

In order to describe the solution space, we introduce the time-independent vector fields  $\Lambda = (\Lambda_1, \dots, \Lambda_7) = (\nabla, \Omega, r\partial_r)$ . Define

$$H_{\Lambda}^m(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) : \Lambda^{\alpha} f \in L^2, |\alpha| \leq m\},$$

with the norm

$$(1.8) \quad \|f\|_{H_{\Lambda}^m} = \sum_{|\alpha| \leq m} \|\Lambda^{\alpha} f\|_{L^2}.$$

Solutions will be constructed in the space  $\dot{H}_{\Gamma}^m(T)$  obtained by closing the set  $C^{\infty}([0, T]; C_0^{\infty}(\mathbb{R}^3))$  in the norm  $\sup_{0 \leq t < T} E_m^{1/2}(u(t))$ . Thus,

$$\dot{H}_{\Gamma}^m(T) \subset \left\{ u(t, x) : \partial u(t, \cdot) \in \bigcap_{j=0}^{m-1} C^j([0, T]; H_{\Lambda}^{m-1-j}) \right\}.$$

By Sobolev embedding, it follows that  $\dot{H}_{\Gamma}^m(T) \subset C^{m-2}([0, T] \times \mathbb{R}^3)$ .

An important intermediate role will be played by the following two weighted norms

$$(1.9) \quad \mathcal{X}_m(u(t)) = \sum_{I=1}^M \sum_{|\alpha|=2} \sum_{|\beta| \leq m-2} \|\langle c_I t - r \rangle \partial^\alpha \Gamma^\beta u^I(t)\|_{L^2(\mathbb{R}^3)},$$

and

$$(1.10) \quad \mathcal{J}_m(u(t)) = \sum_{I=1}^M \sum_{|\alpha| \leq m-1} \left\| r^{-1/2+\mu} \langle r \rangle^{-\mu'} \left( |\partial \Gamma^\alpha u^I(t)| + \frac{\Gamma^\alpha u^I(t)}{r} \right) \right\|_{L^2(\mathbb{R}^3)}^2$$

with  $\mu \in (0, 1/2)$  and  $\mu' > \mu$  to be determined later (the choice will be  $\mu = 1/4$  and  $\mu' = \min(2\rho - 1, 3)/4$ , see (3.1), (3.5)). The second norm is extracted from the local energy norm (also known as KSS-type estimates), which is defined as

$$(1.11) \quad LE_m(t) = \int_0^t \mathcal{J}_m(u(\tau)) d\tau.$$

Let us now state our main result precisely.

**Theorem 1.1.** *Let  $\rho > 1$  and  $\delta \ll 1$ . Assume that the nonlinear terms in (1.4) satisfy the symmetric and null conditions (1.5), (1.6). Then there exist constants  $\varepsilon \ll 1 \ll C_0$ , such that the Cauchy problem for (1.3) has a unique global solution  $u \in \dot{H}_1^\kappa(T)$  for every  $T > 0$ , when the initial data*

$$\partial u(0) \in H_\Lambda^{\kappa-1}(\mathbb{R}^3), \quad \kappa \geq 9$$

satisfying

$$(1.12) \quad E_{\kappa-2}^{1/2}(u(0)) \exp C_0 E_\kappa^{1/2}(u(0)) < \varepsilon.$$

Moreover, the solution satisfies the bounds for some  $C_1, C_2 \geq 1$ ,

$$LE_{\kappa-2}(t) + E_{\kappa-2}(u(t)) < 2C_1 \varepsilon \quad \text{and} \quad LE_\kappa(t) + E_\kappa(u(t)) \leq C_2 E_\kappa(u(0)) \langle t \rangle^{C_2 \varepsilon}.$$

To conclude the introduction, let us give some remarks and comments.

*Remark 1.* We remark here that, the situation for the null form quasilinear problems seems much more delicate than the general quasilinear problem, technically due to the occurrence of the scaling vector field in our argument. Actually, it is not hard to see that we can prove the almost global existence (and global existence for higher dimension) for the solutions to the quasilinear quadratic equations, on asymptotically flat manifolds with small time-dependent metric perturbation, by combining our argument with the approach of Metcalfe and Sogge [41], where there is no need to use the scaling vector field. See Appendix 5 for the proof. Although there is work [22] dealing with null form problems without using the scaling vector field in the literature, it seems difficult to be adapted for the setting of time-dependent asymptotically flat perturbation (except the case of compact perturbation, see [23]).

*Remark 2.* The case  $n = 2$  seems more difficult to handle, mainly because of the lack of the local energy estimates with variable coefficients. In principle, we should be able to prove global existence under the null condition of Alinhac [2], provided that we have such local energy estimates, by combining our argument and that of Hidano [8].

*Remark 3.* The same argument can yield global results for the system with repeated speeds (including the relativistic case), by strengthening the null condition to be nonresonant interaction among the waves with the same wave speeds. See [46] or Section 5 of Chapter II of [48].

*Remark 4.* One may wonder if the standard local energy estimates with variable coefficients in [41] (with  $\mu = 1/2$  instead of  $1/4$ ) is sufficient in our argument. It seems to us that it is not the case, see e.g. (3.2) and (3.5). In this sense, it is the work of [9] which makes the adaption of Sideris and Tu's argument possible.

## 2. PRELIMINARIES

**2.1. Commutation and null forms.** In preparation for the energy estimates, we need to consider the commutation properties of the vector fields  $\Gamma$  with respect to the nonlinear terms. It is necessary to verify that the null structure is preserved upon differentiation, in some sense.

**Lemma 2.1.** *Let  $u$  be a solution of (1.3). Assume that the null condition (1.6) holds for the nonlinearity in (1.4). Then for any  $\alpha \in \mathbb{N}^8$ ,*

$$\square_g \Gamma^\alpha u = \sum_{\beta+\gamma+\mu=\alpha} N_\mu^\alpha(\Gamma^\beta u, \Gamma^\gamma u) + \sum_{|\beta| \leq |\alpha|-1} \left( r_0 \nabla^2 \Gamma^\beta u + r_1 \nabla \Gamma^\beta u \right),$$

in which each  $N_\mu^\alpha$  is a quadratic nonlinearity of the form (1.4) satisfying (1.6), and  $r_m$  with  $m \in \mathbb{N}$  denote functions such that

$$|\nabla^\alpha r_m| \leq C_\alpha \delta \langle r \rangle^{-\rho-m-|\alpha|} \quad \text{for any } \alpha \in \mathbb{N}^3.$$

Moreover, if  $|\mu| = 0$ , then  $N_\mu^\alpha = N$ .

*Proof.* Define

$$[\Gamma, N](u, v) = \Gamma N(u, v) - N(\Gamma u, v) - N(u, \Gamma v).$$

This is a quadratic nonlinearity of the form (1.4). Moreover, by the proof of Lemma 4.1 in [46], we know that  $[\Gamma, N]$  is null form for each  $\Gamma$ .

By (H1), we have

$$\square_g = \square - r_0 \nabla^2 - r_1 \nabla,$$

with  $\square = \text{Diag}(\partial_t^2 - c_1^2 \Delta, \dots, \partial_t^2 - c_M^2 \Delta)$ . Recall that

$$[\square, \Gamma_j] = 2\delta_{j7}\square,$$

we want to prove the result by induction. It is clear the result is true for  $|\alpha| = 0$ . Now assume that it is true for any  $\alpha$  with  $|\alpha| = m$ . Given  $\alpha_0$  with

$|\alpha_0| = m + 1$ , we can find some  $j$  and  $\alpha$  with  $|\alpha| = m$  and  $\Gamma^{\alpha_0} = \Gamma_j \Gamma^\alpha$ . Then by the inductual assumption, we can calculate as follows

$$\begin{aligned}
\Box_g \Gamma^{\alpha_0} u &= \Box_g \Gamma_j \Gamma^\alpha u \\
&= [\Box_g, \Gamma_j] \Gamma^\alpha u + \Gamma_j \Box_g \Gamma^\alpha u \\
&= 2\delta_{j7} \Box \Gamma^\alpha u - [r_0 \nabla^2 + r_1 \nabla, \Gamma_j] \Gamma^\alpha u \\
&\quad + \sum_{\beta+\gamma+\mu=\alpha} \Gamma_j N_\mu^\alpha(\Gamma^\beta u, \Gamma^\gamma u) + \sum_{|\beta| \leq |\alpha|-1} \Gamma_j (r_0 \nabla^2 \Gamma^\beta u + r_1 \nabla \Gamma^\beta u) \\
&= 2\delta_{j7} \Box_g \Gamma^\alpha u + \sum_{|\beta| \leq |\alpha|} (r_0 \nabla^2 \Gamma^\beta u + r_1 \nabla \Gamma^\beta u) \\
&\quad + \sum_{\beta+\gamma+\mu=\alpha} \left\{ [\Gamma_j, N_\mu^\alpha](\Gamma^\beta u, \Gamma^\gamma u) + N_\mu^\alpha(\Gamma_j \Gamma^\beta u, \Gamma^\gamma u) \right. \\
&\quad \left. + N_\mu^\alpha(\Gamma^\beta u, \Gamma_j \Gamma^\gamma u) \right\}
\end{aligned}$$

which is of the required form. This completes the proof.  $\square$

**2.2. Estimates for null forms.** The utility of the null condition is captured in the next lemma, where we get some additional decay in nonlinearities with the null structure (1.6).

**Lemma 2.2.** *Suppose that the nonlinear form  $N(u, v)$  defined (1.4) satisfies the null condition (1.6). For any  $u, v, w \in C^2([0, T] \times \mathbb{R}^3)$  and  $r \geq C_I t/2$ , we have at any point  $(t, x) \in [0, T] \times \mathbb{R}^3$*

$$\begin{aligned}
(2.1a) \quad &|Q_{II}^{I, \alpha\beta\gamma} \partial_\alpha u \partial_\beta \partial_\gamma v| \\
&\leq \frac{C}{\langle r \rangle} \left[ |\Gamma u| |\partial^2 v| + |\partial u| |\partial \Gamma v| + |\partial u| |\partial v| + \langle c_I t - r \rangle |\partial u| |\partial^2 v| \right],
\end{aligned}$$

$$\begin{aligned}
(2.1b) \quad &|Q_{II}^{I, \alpha\beta\gamma} \partial_\alpha u \partial_\beta v \partial_\gamma w| \\
&\leq \frac{C}{\langle r \rangle} \left[ |\Gamma u| |\partial v| |\partial w| + |\partial u| |\Gamma v| |\partial w| + |\partial u| |\partial v| |\Gamma w| \right. \\
&\quad \left. + \langle c_I t - r \rangle |\partial u| |\partial v| |\partial w| \right],
\end{aligned}$$

and

$$(2.1c) \quad |S_{II}^{I, \alpha\beta} \partial_\alpha u \partial_\beta v| \leq \frac{C}{\langle r \rangle} \left[ |\Gamma u| |\partial v| + |\partial u| |\Gamma v| + \langle c_I t - r \rangle |\partial u| |\partial v| |\partial w| \right].$$

*Proof.* The inequalities (2.1a)-(2.1b) are exactly Lemma 5.1 of [46]. The proof of (2.1c) is similar. See also Lemma 5.4 of Chapter II in [48].  $\square$

**2.3. Sobolev-type inequalities.** The following Sobolev inequalities do not involve the Lorentz boost operators. The weight  $\langle c_I t - r \rangle$  compensates for this. We use the notation defined in (1.7), (1.9).

**Lemma 2.3.** *We have the following inequalities for smooth functions  $u : \mathbb{R}_+^{3+1} \rightarrow \mathbb{R}^M$ ,*

$$(2.2) \quad \langle r \rangle^{1/2} |u(t, x)| \leq C E_2^{1/2}(u(t)),$$

$$(2.3) \quad \langle r \rangle |\partial u(t, x)| \leq C E_3^{1/2}(u(t)),$$

$$(2.4) \quad \langle r \rangle \langle c_I t - r \rangle^{1/2} |\partial u^I(t, x)| \leq C \left[ E_3^{1/2}(u(t)) + \mathcal{X}_3(u(t)) \right],$$

$$(2.5) \quad \langle r \rangle \langle c_I t - r \rangle |\partial^2 u^I(t, x)| \leq C \mathcal{X}_4(u(t)),$$

$$(2.6) \quad \langle r \rangle^{1/2} \langle c_I t - r \rangle |\partial u^I(t, x)| \leq C \left[ E_2^{1/2}(u(t)) + \mathcal{X}_3(u(t)) \right].$$

See Lemma 6.1 in [46] for the proof of (2.2)-(2.5) (see also [30], [45]). The inequality (2.6) is just (4.22) of [8]. See also Lemma 5.2 of Chapter II in [48].

Let us give the proof of (2.6) here. Recall the following trace estimate

$$r^{1/2} \|f(r\omega)\|_{L_\omega^4} \leq C \|f\|_{\dot{H}^1}$$

(see (3.16) of Sideris [45] and also (1.3) of [7] for more general version), then

$$\begin{aligned} r^{1/2} \langle c_I t - r \rangle |\partial u^I(t, x)| &\leq C \sum_{|\alpha| \leq 1} r^{1/2} \|\langle c_I t - r \rangle \Omega^\alpha \partial u^I(t, r\omega)\|_{L_\omega^4} \\ &\leq C \sum_{|\alpha| \leq 1} \|\langle c_I t - r \rangle \Omega^\alpha \partial u^I(t, x)\|_{\dot{H}^1} \\ &\leq C \left[ E_2^{1/2}(u(t)) + \mathcal{X}_3(u(t)) \right], \end{aligned}$$

which is exactly (2.6) for  $r \geq 1$ . For the remained case  $r \leq 1$ , we can use a cut-off function argument to derive the required estimate as follows. Let  $\chi$  be a cut-off function such that  $\chi = 1$  for  $|x| \leq 1$  and  $\chi(x) = 0$  for  $|x| \geq 2$ , then for  $r \leq 1$ ,

$$\begin{aligned} &\langle r \rangle^{1/2} \|\langle c_I t - r \rangle \partial u^I(t, x)\|_{L^\infty(|x| \leq 1)} \\ &\leq C \langle t \rangle \|\chi \partial u^I(t, x)\|_{L^\infty} \\ &\leq C \langle t \rangle \|\chi \partial u^I(t, x)\|_{H^2} \\ &\leq C \langle t \rangle \left( \sum_{i=1}^2 \|\nabla^i \partial u^I(t, x)\|_{L^2(|x| \leq 2)} + \|\partial u^I(t, x)\|_{L^\infty(1 < |x| < 2)} \right) \\ &\leq C \sum_{i=1}^2 \|\langle c_I t - r \rangle \nabla^i \partial u^I(t, x)\|_{L^2} + C \|r^{1/2} \langle c_I t - r \rangle \partial u^I(t, x)\|_{L_x^\infty} \\ &\leq C \left[ E_2^{1/2}(u(t)) + \mathcal{X}_3(u(t)) \right], \end{aligned}$$



where we have used (2.6) for  $r \geq 1$  in the last inequality.

**2.4. Local energy estimates.** One of the main extra steps in our proof is to exploit the local energy estimates (also known as Morawetz estimates, KSS estimates), to handle the extra terms arising from the non-flat metric.

**Lemma 2.4.** *Let  $f_0 = (r/(1+r))^{2\mu}$ ,  $f_k = r/(r+2^k)$  with  $k \geq 1$ , and  $u$  be the solution to the equation  $(\partial_t^2 - c_I^2 \Delta + h^{I,\alpha\beta}(t,x)\partial_\alpha\partial_\beta)u^I = F^I$  in  $[0, T] \times \mathbb{R}^n$  with  $h^{I,\alpha\beta} = h^{I,\beta\alpha}$ ,  $\sum_{I=1}^M \sum_{0 \leq \alpha, \beta \leq n} |h^{I,\alpha\beta}| \leq 1/2$  and  $n \geq 3$ , then there exists a constant  $C > 0$ , depending only on the dimension  $n$ , such that*

$$\begin{aligned}
& \sup_{0 \leq t \leq T} E_1(u(t)) + LE_1(T) + (\log(2+T))^{-1} KSS_1(T) \\
& \leq CE_1(u(0)) + C \int_0^T \int_{\mathbb{R}^n} |\partial h| |\partial u|^2 dx dt \\
& \quad + C \left| \sum_{I=1}^M \int_0^T \int_{\mathbb{R}^n} F^I \partial_t u^I dx dt \right| \\
& \quad + C \sup_{k \geq 0} \left| \sum_{I=1}^M \int_0^T \int_{\mathbb{R}^n} f_k \left( \partial_r u^I + \frac{n-1}{2r} u^I \right) F^I dx dt \right| \\
& \quad + C \sup_{k \geq 0} \int_0^T \int_{\mathbb{R}^n} \left\{ \left[ f_k |\partial h| + \left( |f'_k| + \frac{f_k}{r} \right) |h| \right] |\partial u| \left( |\partial u| + \frac{|u|}{r} \right) \right\} dx dt \\
& \leq CE_1(u(0)) + C \int_0^T \int_{\mathbb{R}^n} \left\{ \left( |\partial h| + \frac{|h|}{r^{1-2\mu} \langle r \rangle^{2\mu}} \right) |\partial u| \left( |\partial u| + \frac{|u|}{r} \right) \right\} dx dt \\
& \quad + C \left| \sum_{I=1}^M \int_0^T \int_{\mathbb{R}^n} F^I \partial_t u^I dx dt \right| \\
& \quad + C \sup_{k \geq 0} \left| \sum_{I=1}^M \int_0^T \int_{\mathbb{R}^n} f_k \left( \partial_r u^I + \frac{n-1}{2r} u^I \right) F^I dx dt \right|.
\end{aligned}$$

See Section 2 of Hidano, Wang and Yokoyama [9] for the proof (see also [10], [41] and references therein).

**2.5. Weighted decay estimates.** One of the main steps is to control the weighted norm  $\mathcal{X}_\kappa(u(t))$ . This will be accomplished in this subsection by a type of bootstrap argument, similar to what in [46].

**Lemma 2.5** (Klainerman-Sideris estimate). *Let  $u \in \dot{H}_\Gamma^\kappa(T)$ ,  $\delta \ll 1$  and  $\rho \geq 1$ . Then*

$$(2.7) \quad \mathcal{X}_\kappa(u(t)) \leq C \left[ E_\kappa^{1/2}(u(t)) + \sum_{|\alpha| \leq \kappa-2} \|(t+r) \square_g \Gamma^\alpha u(t)\|_{L^2} \right].$$

*Proof.* The same estimate is known to be true for the standard D'Alembertian  $\square$  instead of  $\square_g$ , see Lemma 7.1 in [46], see also [32] and Lemma 5.3 of

Chapter II in [48]. To complete the proof, we need only to control the norm involving  $\square$  by that of  $\square_{\mathfrak{g}}$  and good terms, as follows. For any  $\alpha \in \mathbb{N}^8$  with  $|\alpha| \leq \kappa - 2$ ,

$$\begin{aligned}
& \|(t+r)\square\Gamma^\alpha u(t)\|_{L^2} - \|(t+r)\square_{\mathfrak{g}}\Gamma^\alpha u(t)\|_{L^2} \\
& \leq \|(t+r)r_0\nabla^2\Gamma^\alpha u(t)\|_{L^2} + \|(t+r)r_1\nabla\Gamma^\alpha u(t)\|_{L^2} \\
& \leq \sum_I (\|\langle c_I t - r \rangle r_{-1}\nabla^2\Gamma^\alpha u^I(t)\|_{L^2} + \|\langle c_I t - r \rangle r_0\nabla\Gamma^\alpha u^I(t)\|_{L^2}) \\
& \leq C\delta\mathcal{X}_\kappa(u(t)) + C\delta\|\nabla(\langle c_I t - r \rangle\nabla\Gamma^\alpha u^I(t))\|_{L^2} \\
& \leq C\delta\mathcal{X}_\kappa(u(t)) + C\delta\|\nabla\Gamma^\alpha u(t)\|_{L^2} \\
& \leq C\delta\mathcal{X}_\kappa(u(t)) + C\delta E_\kappa^{1/2}(u(t)) ,
\end{aligned}$$

where we have used the elementary inequality

$$(2.8) \quad t+r \leq C\langle c_I t - r \rangle \langle r \rangle$$

in the second inequality, the Hardy's inequality and  $\rho \geq 1$  in the third inequality, and the fact that  $|\nabla\langle c_I t - r \rangle| \leq C$  in the fourth inequality.  $\square$

Now we assume that  $u$  solves the nonlinear equation (1.3).

**Lemma 2.6.** *Let  $\rho \geq 1$  and  $u \in \dot{H}_\Gamma^\kappa(T)$  be a solution of (1.3). Define  $\kappa' = \lceil \frac{\kappa-1}{2} \rceil + 3$ . Then for all  $|\alpha| \leq \kappa - 2$ ,*

$$\begin{aligned}
(2.9) \quad \|(t+r)\square_{\mathfrak{g}}\Gamma^\alpha u(t)\|_{L^2} & \leq C[\mathcal{X}_{\kappa'}(u(t))E_{\kappa-1}^{1/2}(u(t)) + \mathcal{X}_\kappa(u(t))E_{\kappa'}^{1/2}(u(t))] \\
& \quad + CE_{\kappa'}^{1/2}(u(t))E_{\kappa-1}^{1/2}(u(t)) + C\delta\mathcal{X}_{\kappa-1}(u(t)) + C\delta E_{\kappa-2}^{1/2}(u(t)).
\end{aligned}$$

*Proof.* There are similar estimates for the case of  $\square$  in Lemma 7.2 of [46] and Lemma 5.2 of [8]. In view of Lemma 2.1, we need to control the terms of the form

$$\begin{aligned}
& \|(t+r)\partial\Gamma^\beta u^I(t)\partial^2\Gamma^\gamma u^J(t)\|_{L^2} , \\
& \|(t+r)\partial\Gamma^\beta u^I(t)\partial\Gamma^\gamma u^J(t)\|_{L^2} ,
\end{aligned}$$

with  $|\beta| + |\gamma| \leq |\alpha| \leq \kappa - 2$  and

$$\|(t+r)r_0\nabla^2\Gamma^\beta u(t)\|_{L^2}^2 + \|(t+r)r_1\nabla\Gamma^\beta u(t)\|_{L^2}$$

with  $|\beta| \leq |\alpha| - 1 \leq \kappa - 3$ . For the first set of terms, we separate two cases: either  $|\beta| \leq \kappa' - 3$  or  $|\gamma| \leq \kappa' - 4$ . In the case of  $|\beta| \leq \kappa' - 3$ , using (2.8) and (2.3), we have

$$\begin{aligned}
\|(t+r)\partial\Gamma^\beta u^I(t)\partial^2\Gamma^\gamma u^J(t)\|_{L^2} & \leq C\|\langle r \rangle\partial\Gamma^\beta u^I(t)\|_{L^\infty}\|\langle c_J t - r \rangle\partial^2\Gamma^\gamma u^J(t)\|_{L^2} \\
& \leq CE_{\kappa'}^{1/2}(u(t))\mathcal{X}_\kappa(u(t)) .
\end{aligned}$$

In the second case, using (2.8) and (2.5), we get

$$\begin{aligned}
\|(t+r)\partial\Gamma^\beta u^I(t)\partial^2\Gamma^\gamma u^J(t)\|_{L^2} & \leq C\|\partial\Gamma^\beta u^I(t)\|_{L^2}\|\langle r \rangle\langle c_J t - r \rangle\partial^2\Gamma^\gamma u^J(t)\|_{L^\infty} \\
& \leq CE_{\kappa-1}^{1/2}(u(t))\mathcal{X}_{\kappa'}(u(t)) .
\end{aligned}$$

Turning to the second set of terms, without loss of generality, we can assume  $|\beta| \leq |\gamma|$  (and so  $|\beta| \leq \kappa' - 3$ ). Then, using (2.6) and (2.5), we get

$$\begin{aligned} & \| (t+r) \partial \Gamma^\beta u^I(t) \partial \Gamma^\gamma u^J(t) \|_{L^2} \\ & \leq C (\| \langle c_I t - r \rangle \partial \Gamma^\beta u^I(t) \|_{L^\infty} + \| \langle r \rangle \partial \Gamma^\beta u^I(t) \|_{L^\infty}) \| \partial \Gamma^\gamma u^J(t) \|_{L^2} \\ & \leq C (E_{\kappa'-1}^{1/2}(u(t)) + \mathcal{X}_{\kappa'}(u(t))) E_{\kappa-1}^{1/2}(u(t)) + C E_{\kappa'}^{1/2}(u(t)) E_{\kappa-1}^{1/2}(u(t)) . \end{aligned}$$

Since  $\rho \geq 1$  and  $|\beta| \leq |\alpha| - 1 = \kappa - 3$ , we see that

$$\begin{aligned} & \| (t+r) r_0 \nabla^2 \Gamma^\beta u^I(t) \|_{L^2} + \| (t+r) r_1 \nabla \Gamma^\beta u^I(t) \|_{L^2} \\ & \leq C \| \langle c_I t - r \rangle r_{-1} \nabla^2 \Gamma^\beta u^I(t) \|_{L^2} + \| \langle c_I t - r \rangle r_0 \nabla \Gamma^\beta u^I(t) \|_{L^2}^2 \\ & \leq C \delta \mathcal{X}_{\kappa-1}(u(t)) + C \delta \| \nabla (\langle c_I t - r \rangle \nabla \Gamma^\beta u(t)) \|_{L^2} \\ & \leq C \delta \mathcal{X}_{\kappa-1}(u(t)) + C \delta E_{\kappa-2}^{1/2}(u(t)) . \end{aligned}$$

This completes the proof.  $\square$

The next result gains control of the weighted norm  $\mathcal{X}$  by the energy. We distinguish two different energies, the smaller of which will remain small. In Section 4, we will allow the larger energy to grow in time.

**Lemma 2.7.** *Let  $u \in \dot{H}_\Gamma^\kappa(T)$ ,  $\kappa \geq 8$ , be a solution of (1.3). Define  $\eta = \kappa - 2$ , and assume that*

$$\delta \ll 1, \quad \varepsilon_0 \equiv \sup_{0 \leq t < T} E_\eta^{1/2}(u(t)) \ll 1, \quad \rho \geq 1.$$

Then for  $0 \leq t < T$ ,

$$(2.10a) \quad \mathcal{X}_\eta(u(t)) \leq C E_\eta^{1/2}(u(t))$$

and

$$(2.10b) \quad \mathcal{X}_\kappa(u(t)) \leq C E_\kappa^{1/2}(u(t)).$$

*Proof.* Let  $\eta' = \left\lceil \frac{\eta-1}{2} \right\rceil + 3$ ,  $\eta = \kappa - 2$ . Since  $\eta \geq 6$ , we have  $\eta' \leq \eta$ . Thus, by Lemmas 2.5 and 2.6, we get from our assumption

$$\begin{aligned} \mathcal{X}_\eta(u(t)) & \leq C [E_\eta^{1/2}(u(t)) + E_{\eta'}^{1/2} \mathcal{X}_\eta + \mathcal{X}_{\eta'} E_{\eta-1}^{1/2} + E_{\eta'}^{1/2} E_{\eta-1}^{1/2}] \\ & \quad + C \delta [\mathcal{X}_{\eta-1}(u(t)) + E_{\eta-2}^{1/2}(u(t))] \\ & \leq C [E_\eta^{1/2}(u(t)) + (\varepsilon_0 + \delta) \mathcal{X}_\eta(u(t))] \end{aligned}$$

Thus, if  $\varepsilon_0$  and  $\delta$  are small enough, the bound (2.10a) results.

Again since  $\kappa \geq 8$ , we have  $\kappa' = \left\lceil \frac{\kappa-1}{2} \right\rceil + 3 \leq \eta = \kappa - 2$ . From Lemmas 2.5 and 2.6, we have

$$\begin{aligned} \mathcal{X}_\kappa(u(t)) & \leq C [E_\kappa^{1/2}(u(t)) + \mathcal{X}_\eta E_{\kappa-1}^{1/2} + \mathcal{X}_\kappa E_\eta^{1/2} + E_\eta^{1/2} E_{\kappa-1}^{1/2}] \\ & \quad + C \delta [\mathcal{X}_{\kappa-1}(u(t)) + E_{\kappa-2}^{1/2}(u(t))] \\ & \leq C (1 + \mathcal{X}_\eta + E_\eta^{1/2} + \delta) E_\kappa^{1/2}(u(t)) + C (E_\eta^{1/2} + \delta) \mathcal{X}_\kappa(u(t)) \end{aligned}$$

If we apply (2.10a),  $\epsilon_0 \ll 1$  and  $\delta \ll 1$ , then

$$\mathcal{X}_\kappa(u(t)) \leq CE_\kappa^{1/2}(u(t)) + C(\epsilon_0 + \delta)\mathcal{X}_\kappa(u(t)),$$

from which (2.10b) follows.  $\square$

### 3. ENERGY AND LOCAL ENERGY ESTIMATES

Assume that  $u(t) \in \dot{H}_\Gamma^\kappa(T)$  is a local solution of the initial value problem for (1.3) (in which we need the symmetry condition (1.5)). Our task will be to show that  $E_\kappa(u(t))$  remains finite for all  $t \geq 0$ . To do so, we will derive a pair of coupled integral inequalities for  $E_\kappa(u(t)) + LE_\kappa(t)$  and  $E_\eta(u(t)) + LE_\eta(t)$ , with  $\eta = \kappa - 2$ . If (1.12) holds with  $2C_1\epsilon \leq \epsilon_0$  occurred in Lemma 2.7, then  $E_\eta^{1/2}(u(0)) < \epsilon$  and  $E_\eta^{1/2}(u(t)) < 2C_1\epsilon$  for certain small interval  $t \in [0, T]$ . Define

$$T_0 = \sup\{T : E_\eta^{1/2}(u(t)) \leq 2C_1\epsilon, t \in [0, T]\}.$$

Here the constants  $C_0, C_1 \geq 1$  will be determined later. All of the following computations will be valid on the time interval  $[0, T_0)$ .

To complete the proof of Theorem 1.1, we need to obtain estimates for  $E_m(u(t))$ , as well as  $\mathcal{I}_m(u(t))$  with  $m = \eta, \kappa$ . Instead of giving the estimates for  $\mathcal{I}_m(u(t))$  directly, we will give the estimates of  $LE_m(u(t))$ .

Since we have

$$(\square_g)^I = \partial_t^2 - c_I^2 \Delta - c_I^2(g^{ij} - \delta^{ij})\partial_i \partial_j + r_1 \nabla,$$

it is easy to see that Lemma 2.4 holds for  $(\square)^I - h^{I,ij}\partial_i \partial_j$  with  $h^{I,ij} = c_I^2(g^{ij} - \delta^{ij})$ . By Lemma 2.1, we see that

$$\begin{aligned} & (\partial_t^2 - c_I^2 \Delta - c_I^2(g^{ij} - \delta^{ij})\partial_i \partial_j)\Gamma^\alpha u^I \\ &= (\square_g \Gamma^\alpha u)^I - r_1 \nabla \Gamma^\alpha u^I \\ &= \sum_{\beta+\gamma+\mu=\alpha} N_\mu^\alpha(\Gamma^\beta u, \Gamma^\gamma u) + \left[ \sum_{|\beta| \leq |\alpha|-1} (r_0 \nabla^2 \Gamma^\beta u + r_1 \nabla \Gamma^\beta u) - r_1 \nabla \Gamma^\alpha u \right] \\ &= F_\alpha + G_\alpha. \end{aligned}$$

Since  $\rho > 1$ , if we assume

$$(3.1) \quad 2\mu' \leq \rho - 1 + 2\mu,$$

then Lemma 2.4 tells us that

$$\begin{aligned}
(3.2) \quad & \sup_{t \in [0, T]} E_m(u(t)) + LE_m(T) \\
& \leq CE_m(u(0)) \\
& \quad + C\delta \sum_{|\alpha| \leq m-1} \int_0^T \int_{\mathbb{R}^3} \left\{ r^{-1+2\mu} \langle r \rangle^{-2\mu-\rho} |\partial \Gamma^\alpha u| \left( |\partial \Gamma^\alpha u| + \frac{|\Gamma^\alpha u|}{r} \right) \right\} dx dt \\
& \quad + C \sum_{|\alpha| \leq m-1} \left| \sum_{I=1}^M \int_0^T \int_{\mathbb{R}^3} F_\alpha^I \partial_t \Gamma^\alpha u^I dx dt \right| \\
& \quad + C \sum_{|\alpha| \leq m-1} \sup_{k \geq 0} \left| \sum_{I=1}^M \int_0^T \int_{\mathbb{R}^3} f_k \left( \partial_r \Gamma^\alpha u^I + \frac{1}{r} \Gamma^\alpha u^I \right) F_\alpha^I dx dt \right| \\
& \quad + C \sum_{|\alpha| \leq m-1} \int_0^T \int_{\mathbb{R}^3} \left( |\partial \Gamma^\alpha u| + \frac{1}{r} |\Gamma^\alpha u| \right) |G_\alpha| dx dt \\
& \leq CE_m(u(0)) + C\delta LE_m(T) \\
& \quad + C \sum_{|\alpha| \leq m-1} \left| \int_0^T \int_{\mathbb{R}^3} \sum_{I=1}^M F_\alpha^I \partial_t \Gamma^\alpha u^I dx dt \right| \\
& \quad + C \sum_{|\alpha| \leq m-1} \sup_{k \geq 0} \left| \int_0^T \int_{\mathbb{R}^3} \sum_{I=1}^M f_k \left( \partial_r \Gamma^\alpha u^I + \frac{1}{r} \Gamma^\alpha u^I \right) F_\alpha^I dx dt \right| \\
& \equiv CE_m(u(0)) + C\delta LE_m(T) + C \sum_{|\alpha| \leq m-1} \left| \int_0^T C_1^\alpha(t) dt \right| \\
& \quad + C \sum_{|\alpha| \leq m-1} \sup_{k \geq 0} \left| \int_0^T C_{2,k}^\alpha(t) dt \right|.
\end{aligned}$$

**3.1. Estimate for  $C_1^\alpha$ .** Notice that

$$C_1^\alpha = \int_{\mathbb{R}^3} \sum_{I=1}^M F_\alpha^I \partial_t \Gamma^\alpha u^I dx = \sum_{I=1}^M \sum_{\beta+\gamma+\mu=\alpha} \int_{\mathbb{R}^3} N_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u) \partial_t \Gamma^\alpha u^I dx$$

where  $N_\mu^{I,\alpha}$  denotes the  $I$ -th component of  $N_\mu^\alpha$ . Among all of the terms, the cases  $|\alpha| = |\gamma| = m-1$  for  $Q_\mu^\alpha$  are quasilinear terms and we want to use the symmetry condition (1.5) to absorb such terms. When  $|\alpha| = |\gamma| = m-1$ ,

we have  $Q_\mu^\alpha = Q$ ,  $\gamma = \alpha$ . Then an integration by parts argument will yield

$$\begin{aligned}
& \sum_{I=1}^M \int_{\mathbb{R}^3} Q^I(u, \Gamma^\alpha u) \partial_t \Gamma^\alpha u^I(t) dx \\
&= \sum_{I=1}^M \int_{\mathbb{R}^3} Q_{JK}^{I, \beta\mu\gamma} \partial_\beta u^J \partial_{\mu\gamma} \Gamma^\alpha u^K \partial_t \Gamma^\alpha u^I dx \\
&= \sum_{I=1}^M Q_{JK}^{I, \beta\mu\gamma} \int_{\mathbb{R}^3} [\partial_\gamma (\partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_t \Gamma^\alpha u^I) \\
&\quad - \partial_\gamma \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_t \Gamma^\alpha u^I - \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma \partial_t \Gamma^\alpha u^I] dx \\
&= \sum_{I=1}^M Q_{JK}^{I, \beta\mu 0} \partial_t \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_t \Gamma^\alpha u^I dx \\
&\quad - \sum_{I=1}^M Q_{JK}^{I, \beta\mu\gamma} \int_{\mathbb{R}^3} \left[ \partial_\gamma \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_t \Gamma^\alpha u^I + \frac{1}{2} \partial_\beta u^J \partial_t (\partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I) \right] dx \\
&= \sum_{I=1}^M (Q_{JK}^{I, \beta\mu 0} \delta_0^\gamma - \frac{1}{2} Q_{JK}^{I, \beta\mu\gamma}) \partial_t \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I dx \\
&\quad - \sum_{I=1}^M Q_{JK}^{I, \beta\mu\gamma} \int_{\mathbb{R}^3} \left[ \partial_\gamma \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_t \Gamma^\alpha u^I - \frac{1}{2} \partial_\beta \partial_t u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I \right] dx
\end{aligned}$$

where we have used the symmetry of the equation (1.5) in the third step. By introducing the notation

$$(3.3) \quad E_1^\alpha(t) = \sum_{I=1}^M (Q_{JK}^{I, \beta\mu 0} \delta_0^\gamma - \frac{1}{2} Q_{JK}^{I, \beta\mu\gamma}) \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I dx,$$

we see that

$$\begin{aligned}
(3.4) \quad & \sum_{|\alpha| \leq m-1} \left| \int_0^T C_1^\alpha(t) dt \right| \\
& \leq \sum_{I=1}^M \sum_{|\alpha| \leq m-1, \beta+\gamma+\mu=\alpha} \left| \int_0^T \int_{\mathbb{R}^3} S_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u) \partial_t \Gamma^\alpha u^I(t) dx dt \right| \\
& \quad + \sum_{I=1}^M \sum_{|\alpha| \leq m-1, \beta+\gamma+\mu=\alpha, |\gamma| < m-1} \left| \int_0^T \int_{\mathbb{R}^3} Q_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u) \partial_t \Gamma^\alpha u^I(t) dx dt \right| \\
& \quad + \sum_{I=1}^M \sum_{|\alpha|=m-1} \left| \int_0^T \int_{\mathbb{R}^3} Q_{JK}^{I,\beta\mu\gamma} \partial_\gamma \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_t \Gamma^\alpha u^I dx dt \right| \\
& \quad + \sum_{I=1}^M \sum_{|\alpha|=m-1} \left| \int_0^T \int_{\mathbb{R}^3} Q_{JK}^{I,\beta\mu\gamma} \partial_\beta \partial_t u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I dx dt \right| \\
& \quad + \sum_{|\alpha|=m-1} |E_1^\alpha(T) - E_1^\alpha(0)| .
\end{aligned}$$

**3.2. Estimate for  $C_{2,k}^\alpha$ .** Let  $L_k = f_k(\partial_r + \frac{1}{r}) = h_k^i(x) \partial_i + h_k(x)$  with  $h_k(x) = f_k(|x|)/|x|$  and  $h_k^i(x) = x^i h_k(x)$ , we see that

$$C_{2,k}^\alpha = \int_{\mathbb{R}^3} \sum_{I=1}^M L_k \Gamma^\alpha u^I F_\alpha^I dx = \sum_{I=1}^M \sum_{\beta+\gamma+\mu=\alpha} \int_{\mathbb{R}^3} N_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u) L_k \Gamma^\alpha u^I dx .$$

As for  $C_1^\alpha$ , for the case  $|\alpha| = |\gamma| = m-1$ , we have  $Q_\mu^{I,\alpha} = Q$ ,  $\gamma = \alpha$ , and

$$\begin{aligned}
& \sum_{I=1}^M \int_{\mathbb{R}^3} Q^I(u, \Gamma^\alpha u) L_k \Gamma^\alpha u^I(t) dx \\
& = \sum_{I=1}^M \int_{\mathbb{R}^3} Q_{JK}^{I,\beta\mu\gamma} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K L_k \Gamma^\alpha u^I dx \\
& = \sum_{I=1}^M Q_{JK}^{I,\beta\mu 0} \partial_t \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K L_k \Gamma^\alpha u^I dx \\
& \quad - \sum_{I=1}^M Q_{JK}^{I,\beta\mu\gamma} \int_{\mathbb{R}^3} [\partial_\gamma \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K L_k \Gamma^\alpha u^I + \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma L_k \Gamma^\alpha u^I] dx .
\end{aligned}$$

The last term in the above identity can be estimated as follows,

$$\begin{aligned}
& - \sum_{I=1}^M Q_{JK}^{I,\beta\mu\gamma} \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K [(\partial_\gamma h_k^i) \partial_i + h_k^i \partial_\gamma \partial_i + (\partial_\gamma h_k) + h_k \partial_\gamma] \Gamma^\alpha u^I dx \\
& = - \sum_{I=1}^M Q_{JK}^{I,\beta\mu\gamma} \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K [(\partial_\gamma h_k^i) \partial_i + (\partial_\gamma h_k) + h_k \partial_\gamma] \Gamma^\alpha u^I dx \\
& \quad - \frac{1}{2} \sum_{I=1}^M Q_{JK}^{I,\beta\mu\gamma} \int_{\mathbb{R}^3} h_k^i \partial_\beta u^J \partial_i (\partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I) dx \\
& = - Q_{JK}^{I,\beta\mu\gamma} \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K [(\partial_\gamma h_k^i) \partial_i + (\partial_\gamma h_k) + h_k \partial_\gamma] \Gamma^\alpha u^I dx \\
& \quad + \frac{1}{2} Q_{JK}^{I,\beta\mu\gamma} \int_{\mathbb{R}^3} [(\partial_i h_k^i) \partial_\beta u^J + h_k^i \partial_\beta \partial_i u^J] \partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I dx \\
& = - \sum_{I=1}^M Q_{JK}^{I,\beta\mu\gamma} \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \left[ (\partial_\gamma h_k^i) \partial_i + \left( h_k - \frac{1}{2} \partial_i h_k^i \right) \partial_\gamma + \partial_\gamma h_k \right] \Gamma^\alpha u^I dx \\
& \quad + \frac{1}{2} \sum_{I=1}^M Q_{JK}^{I,\beta\mu\gamma} \int_{\mathbb{R}^3} h_k^i \partial_\beta \partial_i u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I dx
\end{aligned}$$

where we have used the symmetry of the equation 1.5. It is easy to check that, by setting, say,  $\mu = 1/4$ , there is a uniform constant  $C > 0$  which is independent of  $k \geq 0$  such that

$$|\nabla h_k^i| + |h_k| \leq C r^{-1/2} \langle r \rangle^{-1/2}, \quad |\nabla h_k| \leq C r^{-3/2} \langle r \rangle^{-1/2}.$$

Moreover, by (2.3), we have

$$|\partial u| \leq C \langle r \rangle^{-1} E_3^{1/2}(u(t)) \leq C C_1 \varepsilon \langle r \rangle^{-1}.$$

Then, the first term in the above expression is controlled by  $C C_1 \varepsilon \mathcal{J}_m(t)$  with

$$(3.5) \quad \mu = 1/4 < \mu' \leq 3/4.$$

In summary, by introducing the notation

$$(3.6) \quad E_{2,k}^\alpha = \sum_{I=1}^M Q_{JK}^{I,\beta\mu 0} \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K L_k \Gamma^\alpha u^I dx,$$



we have proven the following

$$\begin{aligned}
(3.7) \quad & \sum_{|\alpha| \leq m-1} \left| \int_0^T C_{2,k}^\alpha(t) dt \right| \\
& \leq \sum_{I=1}^M \sum_{|\alpha| \leq m-1, \beta+\gamma+\mu=\alpha} \left| \int_0^T \int_{\mathbb{R}^3} S_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u) L_k \Gamma^\alpha u^I(t) dx dt \right| \\
& + \sum_{|\alpha| \leq m-1, \beta+\gamma+\mu=\alpha, |\gamma| < m-1} \left| \int_0^T \int_{\mathbb{R}^3} Q_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u) L_k \Gamma^\alpha u^I(t) dx dt \right| \\
& + \sum_{I=1}^M \sum_{|\alpha|=m-1} \left| \int_0^T \int_{\mathbb{R}^3} Q_{JK}^{I,\beta\mu\gamma} \partial_\gamma \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K L_k \Gamma^\alpha u^I dx dt \right| \\
& + \sum_{I=1}^M \sum_{|\alpha|=m-1} \left| \int_0^T \int_{\mathbb{R}^3} Q_{JK}^{I,\beta\mu\gamma} h_k^i \partial_\beta \partial_i u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I dx dt \right| \\
& + \sum_{|\alpha|=m-1} |E_{2,k}^\alpha(T) - E_{2,k}^\alpha(0)| + CC_1 \varepsilon L E_m(T) .
\end{aligned}$$

#### 4. PROOF OF THE ENERGY INEQUALITY

In this section, we will complete the proof of Theorem 1.1.

At first, we claim that by (3.2), (3.4) and (3.7), we can deduce that for some universal constant  $C_3 \geq 1$ ,

$$\begin{aligned}
(4.1) \quad & L E_\kappa(T) + \sup_{0 \leq t \leq T} E_\kappa(u(t)) \\
& \leq C_3 E_\kappa(u(0)) + C_3 \int_0^T \langle t \rangle^{-1} E_\eta^{1/2}(u(t)) E_\kappa(u(t)) dt \\
& + C_3 C_1 (\delta + \varepsilon) L E_\kappa(T) + C_3 \sum_{|\alpha|=\kappa-1} \sup_{k \geq 0} |E_{2,k}^\alpha(T) - E_{2,k}^\alpha(0)| \\
& + C_3 \sum_{|\alpha|=\kappa-1} |E_1^\alpha(T) - E_1^\alpha(0)| ,
\end{aligned}$$

and

$$\begin{aligned}
(4.2) \quad & L E_\eta(T) + \sup_{0 \leq t \leq T} E_\eta(u(t)) \\
& \leq C_3 E_\eta(u(0)) + C_3 \int_0^T \langle t \rangle^{-1} t^{-1/2} E_\kappa^{1/2}(u(t)) E_\eta(u(t)) dt \\
& + C_3 C_1 (\delta + \varepsilon) L E_\eta(T) + C_3 \sum_{|\alpha|=\eta-1} \sup_{k \geq 0} |E_{2,k}^\alpha(T) - E_{2,k}^\alpha(0)| \\
& + C_3 \sum_{|\alpha|=\eta-1} |E_1^\alpha(T) - E_1^\alpha(0)| .
\end{aligned}$$

Comparing the right hand sides in the estimates (3.4) and (3.7), we find similar terms involving the integral of

$$(4.3) \quad D_1^1 = Q_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u) \partial \Gamma^\alpha u^I(t), \quad D_1^2 = h_k(x) Q_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u) \Gamma^\alpha u^I(t)$$

for  $|\alpha| \leq m-1$ ,  $\beta + \gamma + \mu = \alpha$  and  $|\gamma| < m-1$ ,

$$(4.4) \quad D_2^1 = Q_{JK}^{I,\beta\mu\gamma} \partial_\gamma \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial \Gamma^\alpha u^I, \quad D_2^2 = h_k(x) Q_{JK}^{I,\beta\mu\gamma} \partial_\gamma \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \Gamma^\alpha u^I$$

for  $|\alpha| = m-1$ ,

$$(4.5) \quad D_3^1 = Q_{JK}^{I,\beta\mu\gamma} \partial_\beta \partial u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I, \quad D_3^2 = 0$$

for  $|\alpha| = m-1$ , and the semilinear terms

$$(4.6) \quad D_4^1 = S_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u) \partial \Gamma^\alpha u^I(t), \quad D_4^2 = h_k(x) S_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u) \Gamma^\alpha u^I(t)$$

for  $|\alpha| \leq m-1$ ,  $\beta + \gamma + \mu = \alpha$ .

To prove the claim, we need only to control these terms for  $m = \kappa, \eta$  separately. The proof will be similar to the proof of (8.5) and (8.6) in [46]. Here, we give the proof of the bound for the terms occurred on the right hand side of (3.7), that is, to control

$$I_j = \left| \int_{\mathbb{R}^3} h_k^i(x) D_j^1 dx \right|$$

and

$$II_j = \left| \int_{\mathbb{R}^3} D_j^2 dx \right|$$

with  $j = 1, 2, 3, 4$ . The bound for the other terms follows by the same argument. In principle, the quasilinear and semilinear terms in  $I_j$  has been obtained in [46] and [8] separately. The new terms are the terms in  $II_j$ . Here, we give the proof of both estimates of  $I_j$  and  $II_j$  for completeness.

**4.1. Higher energy.** For the first series of estimates we take  $m = \kappa$  in (4.3)-(4.6). Recall that  $|h_k^i| \leq 1$ ,  $|h_k(x)| \leq r^{-1/2}(1+r)^{-1/2}$  and using Hardy's inequality, we obtain immediately

$$(4.7) \quad \begin{aligned} & \sum_j (I_j + II_j) \\ & \leq C \sum_{I,J=1}^M \sum_{|\alpha| \leq \kappa-1, \beta+\gamma \leq \alpha, |\gamma| \leq \kappa-2} \|\partial \Gamma^\beta u^I \partial^2 \Gamma^\gamma u^J\|_{L^2} \|\partial \Gamma^\alpha u\|_{L^2} \\ & \quad + C \sum_{I,J=1}^M \sum_{|\alpha| \leq \kappa-1, \beta+\gamma \leq \alpha} \|\partial \Gamma^\beta u^I \partial \Gamma^\gamma u^J\|_{L^2} \|\partial \Gamma^\alpha u\|_{L^2}. \end{aligned}$$

For the first term on the right-hand side of (4.7), we have either  $|\beta| \leq \kappa'$  or  $|\gamma| \leq \kappa'-1$ , with  $\kappa' = [\kappa/2]$ . Note that since  $\kappa \geq 9$ , we have  $\kappa'+3 \leq \kappa-2 = \eta$ . We will also use that  $\langle t \rangle \leq C \langle r \rangle \langle c_{Jt} - r \rangle$ .

In the first case, we estimate using (2.3) and (2.10b)

$$\begin{aligned}\|\partial\Gamma^\beta u^I \partial^2 \Gamma^\gamma u^J\|_{L^2} &\leq C\langle t\rangle^{-1}\|\langle r\rangle\partial\Gamma^\beta u^I\|_{L^\infty}\|\langle c_J t - r\rangle\partial^2 \Gamma^\gamma u^J\|_{L^2} \\ &\leq C\langle t\rangle^{-1}E_{|\beta|+3}^{1/2}(u(t))\mathcal{X}_\kappa(u(t)) \\ &\leq C\langle t\rangle^{-1}E_\eta^{1/2}(u(t))E_\kappa^{1/2}(u(t)).\end{aligned}$$

In the second case, we use (2.5) and then (2.10a)

$$\begin{aligned}\|\partial\Gamma^\beta u^I \partial^2 \Gamma^\gamma u^J\|_{L^2} &\leq C\langle t\rangle^{-1}\|\partial\Gamma^\beta u\|_{L^2}\|\langle r\rangle\langle c_J t - r\rangle\partial^2 \Gamma^\gamma u^J\|_{L^\infty} \\ &\leq C\langle t\rangle^{-1}E_\kappa^{1/2}(u(t))\mathcal{X}_{|\gamma|+4}(u(t)) \\ &\leq C\langle t\rangle^{-1}E_\kappa^{1/2}(u(t))\mathcal{X}_\eta(u(t)) \\ &\leq C\langle t\rangle^{-1}E_\kappa^{1/2}(u(t))E_\eta^{1/2}(u(t)).\end{aligned}$$

For the second term on the right-hand side of (4.7), as in the proof of Lemma 2.6, we can use (2.6) and (2.5) to get (assuming  $|\beta| \leq |\gamma|$ )

$$\begin{aligned}&\|\partial\Gamma^\beta u^I(t)\partial\Gamma^\gamma u^J(t)\|_{L^2} \\ &\leq C\langle t\rangle^{-1}(\|\langle c_I t - r\rangle\partial\Gamma^\beta u^I(t)\|_{L^\infty} + \|\langle r\rangle\partial\Gamma^\beta u^I(t)\|_{L^\infty})\|\partial\Gamma^\gamma u^J(t)\|_{L^2} \\ &\leq C\langle t\rangle^{-1}(E_\eta^{1/2}(u(t)) + \mathcal{X}_\eta(u(t)))E_\kappa^{1/2}(u(t)) + CE_\eta^{1/2}(u(t))E_\kappa^{1/2}(u(t)) \\ &\leq C\langle t\rangle^{-1}E_\eta^{1/2}(u(t))E_\kappa^{1/2}(u(t)).\end{aligned}$$

Going back to (4.7), we have established the inequality (4.1).

**4.2. Lower energy.** The second series of energy type estimates with  $m = \eta$  will exploit the null condition. Let  $c_0 = \min\{c_I\}$ , the integrals will be subdivided into separate integrals over the regions  $r \leq c_0 t/2$  and  $r \geq c_0 t/2$ .

**Inside the cones.** On the region  $r \leq c_0 t/2$ , we have that

$$\begin{aligned}\sum_j I_j(r \leq c_0 t/2) &\leq C \sum_{I,J,K} \sum_{\beta+\gamma \leq \alpha, |\alpha| \leq \eta-1, \gamma \leq \eta-2} \|\partial\Gamma^\beta u^I \partial^2 \Gamma^\gamma u^J \partial\Gamma^\alpha u^K\|_{L^1(r \leq c_0 t/2)} \\ &\quad + C \sum_{I,J,K} \sum_{|\alpha| \leq \eta-1, \beta+\gamma \leq \alpha} \|\partial\Gamma^\beta u^I \partial\Gamma^\gamma u^J \partial\Gamma^\alpha u^K\|_{L^1(r \leq c_0 t/2)}.\end{aligned}$$

Since  $r \leq c_0 t/2$ , we have  $\langle t \rangle \leq C\langle c_I t - r \rangle$  for any  $I$ . Recall also  $|\beta| + 3 \leq \kappa$ ,  $|\gamma| + 2 \leq \eta$ , and  $|\alpha| + 1 \leq \eta$ , thus, using (2.4) and Lemma 2.7, the terms in the first part of the right hand side can be estimated by

$$\begin{aligned}&C\langle t\rangle^{-3/2}\|\langle c_I t - r\rangle^{1/2}\partial\Gamma^\beta u^I\langle c_J t - r\rangle\partial^2 \Gamma^\gamma u^J\partial\Gamma^\alpha u^K\|_{L^1(r \leq c_0 t/2)} \\ &\leq C\langle t\rangle^{-3/2}\|\langle c_I t - r\rangle^{1/2}\partial\Gamma^\beta u^I\|_{L^\infty}\|\langle c_J t - r\rangle\partial^2 \Gamma^\gamma u\|_{L^2}\|\partial\Gamma^\alpha u\|_{L^2} \\ &\leq C\langle t\rangle^{-3/2}\left[E_{|\beta|+3}^{1/2}(u(t)) + \mathcal{X}_{|\beta|+3}(u(t))\right]\mathcal{X}_{|\gamma|+2}(u(t))E_\eta^{1/2}(u(t)) \\ &\leq C\langle t\rangle^{-3/2}E_\eta(u(t))E_\kappa^{1/2}(u(t)).\end{aligned}$$

For the terms in the second part, using (2.6) instead of (2.4) and assuming  $|\beta| \leq |\gamma|$  (and so  $|\beta| \leq [\eta - 1]/2 \leq \eta - 3$ ), we see

$$\begin{aligned}
& \|\partial\Gamma^\beta u^I \partial\Gamma^\gamma u^J \partial\Gamma^\alpha u^K\|_{L^1(r \leq c_0 t/2)} \\
& \leq \|\partial\Gamma^\beta u^I \partial\Gamma^\gamma u^J\|_{L^2(r \leq c_0 t/2)} \|\partial\Gamma^\alpha u^K\|_{L^2} \\
& \leq \langle t \rangle^{-3/2} \|r \langle c_I t - r \rangle^{1/2} \partial\Gamma^\beta u^I\|_{L^\infty} \left\| \frac{\langle c_J t - r \rangle}{r} \partial\Gamma^\gamma u^J \right\|_{L^2} E_\eta^{1/2}(u(t)) \\
& \leq \langle t \rangle^{-3/2} \left( E_{|\beta|+2}^{1/2}(u(t)) + \mathcal{X}_{|\beta|+3}(u(t)) \right) \left( E_{|\gamma|+1}^{1/2}(u(t)) + \mathcal{X}_{|\gamma|+2}(u(t)) \right) E_\eta^{1/2}(u(t)) \\
& \leq \langle t \rangle^{-3/2} E_\eta(u(t)) E_\kappa^{1/2}(u(t)),
\end{aligned}$$

where in the third inequality we have used the Hardy inequality.

The estimates for  $II_j(r \leq c_0 t/2)$  proceeds similarly with the obvious modification by using the Hardy inequality for terms involving  $\Gamma^\alpha u$  (and  $|h_k(x)| \leq C r^{-1/2} \langle r \rangle^{-1/2} \leq C/r$ ). This gives us the required upper bound for the portion of the integrals over  $r \leq c_0 t/2$  in (4.2).

**Away from the origin.** It remains to give the estimate for  $r \geq c_0 t/2$ . It is here, finally, where the difference of speed  $c_I$  and the null condition enters.

**Non-resonance.** Let us start with non-resonant terms, that is, those for which  $(I, J, K) \neq (K, K, K)$ . In this case, for  $I_j + II_j(r \geq c_0 t/2)$ , we need to control the quasilinear terms

$$(4.8) \quad \sum_{(I,J) \neq (K,K)} \sum_{\beta+\gamma \leq \alpha, |\alpha| \leq \eta-1, \gamma \leq \eta-2} \left\| \partial\Gamma^\beta u^I \partial^2 \Gamma^\gamma u^J \left( |\partial\Gamma^\alpha u^K| + \frac{|\Gamma^\alpha u^K|}{r^{1/2} \langle r \rangle^{1/2}} \right) \right\|_{L^1(r \geq c_0 t/2)}$$

and the semilinear terms

$$(4.9) \quad \sum_{(I,J) \neq (K,K)} \sum_{|\alpha| \leq \eta-1, \beta+\gamma \leq \alpha} \left\| \partial\Gamma^\beta u^I \partial\Gamma^\gamma u^J \left( |\partial\Gamma^\alpha u^K| + \frac{|\Gamma^\alpha u^K|}{r^{1/2} \langle r \rangle^{1/2}} \right) \right\|_{L^1(r \geq c_0 t/2)}.$$

We separate two cases for (4.8):  $I \neq J$  and  $I = J \neq K$ . In the first case, we have  $c_I \neq c_J$ , and

$$\langle t \rangle^{3/2} \leq C \langle r \rangle (\langle c_I t - r \rangle + \langle c_J t - r \rangle)^{1/2} \leq C \langle r \rangle \langle c_I t - r \rangle^{1/2} \langle c_J t - r \rangle^{1/2}.$$

Using (2.4) and Hardy's inequality we have the estimate

$$\begin{aligned}
& \|\partial\Gamma^\beta u^I \partial^2 \Gamma^\gamma u^J \partial\Gamma^\alpha u^K\|_{L^1(r \geq c_0 t/2)} + \|\partial\Gamma^\beta u^I \partial^2 \Gamma^\gamma u^J \Gamma^\alpha u^K / r\|_{L^1(r \geq c_0 t/2)} \\
& \leq C \langle t \rangle^{-3/2} \|\langle r \rangle \langle c_I t - r \rangle^{1/2} \partial\Gamma^\beta u^I\|_{L^\infty} \|\langle c_J t - r \rangle^{1/2} \partial^2 \Gamma^\gamma u^J\|_{L^2} \\
& \quad \times (\|\partial\Gamma^\alpha u^K\|_{L^2} + \|\Gamma^\alpha u^K / r\|_{L^2}) \\
& \leq C \langle t \rangle^{-3/2} \left[ E_{|\beta|+3}^{1/2}(u(t)) + \mathcal{X}_{|\beta|+3}(u(t)) \right] \mathcal{X}_{|\gamma|+2}(u(t)) E_{|\alpha|+1}^{1/2}(u(t)) \\
& \leq C \langle t \rangle^{-3/2} E_\mu(u(t)) E_\kappa^{1/2}(u(t)).
\end{aligned}$$

Otherwise, if  $I = J \neq K$ , (2.4) and (2.2), we get

$$\begin{aligned}
& \|\partial\Gamma^\beta u^I \partial^2\Gamma^\gamma u^I \partial\Gamma^\alpha u^K\|_{L^1(r \geq c_0 t/2)} \\
& + \|r^{-1/2} \langle r \rangle^{-1/2} \partial\Gamma^\beta u^I \partial^2\Gamma^\gamma u^I \Gamma^\alpha u^K\|_{L^1(r \geq c_0 t/2)} \\
& \leq Ct^{-1/2} \langle t \rangle^{-1} \|\partial\Gamma^\beta u^I\|_{L^2} \|\langle c_I t - r \rangle^{1/2} \partial^2\Gamma^\gamma u^I\|_{L^2} \\
& \quad \times \left( \|\langle r \rangle \langle c_K t - r \rangle^{1/2} \partial\Gamma^\alpha u^K\|_{L^\infty} + \|\langle r \rangle^{1/2} \Gamma^\alpha u^K\|_{L^\infty} \right) \\
& \leq Ct^{-1/2} \langle t \rangle^{-1} E_{|\beta|+1}^{1/2}(u(t)) \mathcal{X}_{|\gamma|+2}(u(t)) \left[ E_{|\alpha|+3}^{1/2}(u(t)) + \mathcal{X}_{|\alpha|+3}(u(t)) \right] \\
& \leq Ct^{-1/2} \langle t \rangle^{-1} E_\mu(u(t)) E_\kappa^{1/2}(u(t)).
\end{aligned}$$

For the semilinear terms (4.9), we have  $I \neq K$  and so  $c_I \neq c_K$ . Assuming  $c_I < c_K$ , then

$$\begin{aligned}
& \left\| \partial\Gamma^\beta u^I \partial\Gamma^\gamma u^J \left( |\partial\Gamma^\alpha u^K| + \frac{|\Gamma^\alpha u^K|}{r^{1/2} \langle r \rangle^{1/2}} \right) \right\|_{L^1(r \geq c_0 t/2)} \\
& \leq Ct^{-1/2} \langle t \rangle^{-1} \left\| \partial\Gamma^\beta u^I \partial\Gamma^\gamma u^J \right\|_{L^1} \|\langle r \rangle^{1/2} \Gamma^\alpha u^K\|_{L^\infty} \\
& \quad + C \langle t \rangle^{-3/2} \left\| \partial\Gamma^\beta u^I \partial\Gamma^\gamma u^J \right\|_{L^1} \|\langle r \rangle \langle c_K t - r \rangle^{1/2} \partial\Gamma^\alpha u^K\|_{L^\infty(c_0 t/2 < r < (c_I + c_K)t/2)} \\
& \quad + C \langle t \rangle^{-3/2} \left\| \partial\Gamma^\alpha u^K \partial\Gamma^\gamma u^J \right\|_{L^1} \|\langle r \rangle \langle c_I t - r \rangle^{1/2} \partial\Gamma^\beta u^I\|_{L^\infty(r \geq (c_I + c_K)t/2)} \\
& \leq Ct^{-1/2} \langle t \rangle^{-1} E_\mu(u(t)) E_\kappa^{1/2}(u(t))
\end{aligned}$$

by using (2.4) and (2.2). The case  $c_I > c_K$  can be handled the same way. This completes the estimates for non-resonant terms over  $r \geq c_0 t/2$  in (4.2).

**Resonance.** In the resonant case, we have  $(I, J, K) = (K, K, K)$  and we will denote  $u^K = u$  and  $c_K = c$ . An application of Lemma 2.2 yields the following upper bound for  $I_j(r \geq c_0 t/2)$ :

$$\begin{aligned}
& C \langle t \rangle^{-1} \sum_{\beta+\gamma \leq \alpha, |\gamma| \leq \eta-2, |\alpha| \leq \eta-1} \left[ \|\Gamma^\beta u \partial^2\Gamma^\gamma u \partial\Gamma^\alpha u\|_{L^1(r \geq c_0 t/2)} \right. \\
& + \|\partial\Gamma^\beta u \partial\Gamma^\gamma u \partial\Gamma^\alpha u\|_{L^1(r \geq c_0 t/2)} \\
& \left. + \|\langle ct - r \rangle \partial\Gamma^\beta u \partial^2\Gamma^\gamma u \partial\Gamma^\alpha u\|_{L^1(r \geq c_0 t/2)} \right] \\
& + C \langle t \rangle^{-1} \sum_{\beta+\gamma \leq \alpha, |\alpha| \leq \eta-1} \left[ \|\Gamma^\beta u \partial\Gamma^\gamma u \partial\Gamma^\alpha u^I\|_{L^1(r \geq c_0 t/2)} \right. \\
& \left. + \|\langle ct - r \rangle \partial\Gamma^\beta u \partial\Gamma^\gamma u \partial\Gamma^\alpha u\|_{L^1(r \geq c_0 t/2)} \right].
\end{aligned}$$

We still need to get an additional decay factor of  $t^{-1/2}$ .

Since  $r \geq c_0 t/2$ , we have  $\langle r \rangle \geq C\langle t \rangle$ . Thus, we have using (2.2)

$$\begin{aligned}
& \|\Gamma\Gamma^\beta u \partial^2 \Gamma^\gamma u \partial \Gamma^\alpha u\|_{L^1(r \geq c_0 t/2)} \\
& \leq C\langle t \rangle^{-1/2} \|\langle r \rangle^{1/2} \Gamma\Gamma^\beta u\|_{L^\infty(r \geq c_0 t/2)} \|\partial^2 \Gamma^\gamma u\|_{L^2} \|\partial \Gamma^\alpha u\|_{L^2} \\
& \leq C\langle t \rangle^{-1/2} E_{|\beta|+3}^{1/2}(u(t)) E_\eta(u(t)) \\
& \leq C\langle t \rangle^{-1/2} E_\kappa^{1/2}(u(t)) E_\eta(u(t)).
\end{aligned}$$

In a similar fashion, the second term is handled using (2.3):

$$\begin{aligned}
& \|\partial \Gamma^\beta u \partial \Gamma^\gamma u \partial \Gamma^\alpha u\|_{L^1(r \geq c_0 t/2)} \\
& \leq C\langle t \rangle^{-1} \|\partial \Gamma^\beta u\|_{L^2} \|\langle r \rangle \partial \Gamma^\gamma u\|_{L^\infty(r \geq c_0 t/2)} \|\partial \Gamma^\alpha u\|_{L^2} \\
& \leq C\langle t \rangle^{-1} E_{|\gamma|+3}^{1/2}(u(t)) E_\eta(u(t)) \\
& \leq C\langle t \rangle^{-1} E_\kappa^{1/2}(u(t)) E_\eta(u(t)).
\end{aligned}$$

The third term is estimated using (2.3) again and (2.10a).

$$\begin{aligned}
& \|\langle ct - r \rangle \partial \Gamma^\beta u \partial^2 \Gamma^\gamma u \partial \Gamma^\alpha u\|_{L^1(r \geq c_0 t/2)} \\
& \leq C\langle t \rangle^{-1} \|\langle r \rangle \partial \Gamma^\beta u\|_{L^\infty(r \geq c_0 t/2)} \|\langle ct - r \rangle \partial^2 \Gamma^\gamma u\|_{L^2} \|\partial \Gamma^\alpha u\|_{L^2} \\
& \leq C\langle t \rangle^{-1} E_{|\beta|+3}^{1/2}(u(t)) \mathcal{X}_{|\gamma|+2}(u(t)) E_\eta^{1/2}(u(t)) \\
& \leq C\langle t \rangle^{-1} E_\kappa^{1/2}(u(t)) E_\eta(u(t)).
\end{aligned}$$

For the terms arising in semilinear part, by (2.2), (2.6) and the fact  $\langle r \rangle \geq C\langle t \rangle$ , we have

$$\begin{aligned}
& \|\Gamma\Gamma^\beta u \partial \Gamma^\gamma u \partial \Gamma^\alpha u^I\|_{L^1(r \geq c_0 t/2)} + \|\langle ct - r \rangle \partial \Gamma^\beta u \partial \Gamma^\gamma u \partial \Gamma^\alpha u\|_{L^1(r \geq c_0 t/2)} \\
& \leq \langle t \rangle^{-1/2} (\|\langle r \rangle^{1/2} \Gamma\Gamma^\beta u\|_{L^\infty} + \|\langle r \rangle^{1/2} \langle ct - r \rangle \partial \Gamma^\beta u\|_{L^\infty}) \|\partial \Gamma^\gamma u\|_{L^2} \|\partial \Gamma^\alpha u\|_{L^2} \\
& \leq \langle t \rangle^{-1/2} (E_{|\beta|+3}^{1/2}(u(t)) + \mathcal{X}_{|\beta|+3}) E_\eta(u(t)) \\
& \leq C\langle t \rangle^{-1/2} E_\kappa^{1/2}(u(t)) E_\eta(u(t)).
\end{aligned}$$

To complete the proof of (4.2), we still need to give the estimate for  $II_j(t)$ . In the resonant situation, noting that we always use  $L^2$  norm to control the terms involving  $\partial \Gamma^\alpha u$  in the proof of  $I_j(t)$ , it is easy to adapt the previous proof to get the required estimate for  $II_j(t)$  by using Hardy's inequality and the fact that  $|h_k(x)| \leq Cr^{-1}$ .

**4.3. Conclusion of the proof.** We are now ready to complete the proof of Theorem 1.1, by using (4.1) and (4.2).

Recall the definition (3.3) and (3.6) for  $E_0^\alpha$ ,  $E_1^\alpha$  and  $E_{2,k}^\alpha$ , we know by Sobolev embedding,

$$\begin{aligned} \sum_{|\alpha|=m-1} (|E_1^\alpha(t)| + \sup_{k \geq 0} |E_{2,k}^\alpha(t)|) &\leq C \|\partial u\|_{L^\infty} E_m(u(t)) \\ &\leq C E_3^{1/2}(u(t)) E_m(u(t)) \\ &\leq C_4 C_1 \varepsilon E_m(u(t)) \end{aligned}$$

for some  $C_4 \geq 1$ , and  $1 \leq m \leq \kappa$ .

Based on this observation and the smallness assumption of  $\varepsilon$  and  $\delta$  (such that  $8C_3C_4C_1(\varepsilon + \delta) \leq 1$ ), we can easily obtain the following inequalities from (4.1) and (4.2),

$$\begin{aligned} (4.10) \quad &LE_\kappa(T) + \sup_{0 \leq t \leq T} E_\kappa(u(t)) \\ &\leq 2C_3 E_\kappa(u(0)) + 2C_3 \int_0^T \langle t \rangle^{-1} E_\eta^{1/2}(u(t)) E_\kappa(u(t)) dt \end{aligned}$$

and

$$\begin{aligned} (4.11) \quad &LE_\eta(T) + \sup_{0 \leq t \leq T} E_\eta(u(t)) \\ &\leq 2C_3 E_\eta(u(0)) + 2C_3 \int_0^T t^{-1/2} \langle t \rangle^{-1} E_\kappa^{1/2}(u(t)) E_\eta(u(t)) dt \end{aligned}$$

on any interval  $[0, T]$  with

$$T < T_0 = \sup\{T : E_\eta^{1/2}(u(t)) \leq 2C_1\varepsilon, t \in [0, T]\}.$$

Since  $E_\eta^{1/2}(u(t)) \leq 2C_1\varepsilon$ , an application of the Gronwall inequality to (4.10) gives us

$$LE_\kappa(t) + E_\kappa(u(t)) \leq 2C_3 e^{4C_3C_1\varepsilon} E_\kappa(u(0)) \langle t \rangle^{4C_3C_1\varepsilon}.$$

Inserting this bound into (4.11), we obtain

$$\begin{aligned} &LE_\eta(t) + E_\eta(u(t)) \leq 2C_3 E_\eta(u(0)) \\ &\times \exp \left( (2C_3)^{3/2} e^{2C_3C_1\varepsilon} E_\kappa^{1/2}(u(0)) \int_0^\infty \langle t \rangle^{-1+2C_3C_1\varepsilon} t^{-1/2} dt \right). \end{aligned}$$

Setting  $C_1 = \sqrt{2C_3}$ ,

$$C_0 = C_1^3 e^{C_1^3\varepsilon} \int_0^\infty \langle t \rangle^{-1+C_1^3\varepsilon} t^{-1/2} dt / 2,$$

and  $C_2 = \max(C_1^2 e^{2C_1^3\varepsilon}, 2C_1^3)$ , then we have

$$LE_\eta(t) + E_\eta(u(t)) \leq 2C_3 E_\eta(u(0)) \exp(2C_0 E_\kappa^{1/2}(u(0))) < 2C_3 \varepsilon^2 = (C_1 \varepsilon)^2,$$

$$LE_\kappa(t) + E_\kappa(u(t)) \leq C_2 E_\kappa(u(0)) \langle t \rangle^{C_2\varepsilon}.$$

To ensure the finiteness of  $C_0$ , in addition to the smallness assumption on  $\varepsilon$  to absorb the perturbation, we need also require  $C_1^3\varepsilon < 1/2$ .

With this we see  $E_\eta^{1/2}(u(t))$  remains less than  $C_1\varepsilon$  throughout the interval  $0 \leq t < T_0$ . A standard continuity argument shows that  $E_\eta(u(t))$  is bounded for all time, which completes the proof of Theorem 1.1.

## 5. APPENDIX: THE CASE OF ASYMPTOTICALLY FLAT MANIFOLDS

In this appendix, we will give the almost global existence (and global existence respectively) for the Cauchy problem of the quasilinear wave system (1.3) when spatial dimension is three (and higher), posed on certain asymptotically flat manifolds.

Let us begin with the space-time manifolds. We consider the asymptotically flat Lorentzian manifolds  $(\mathbb{R}^{1+n}, \mathbf{g})$  with

$$\mathbf{g} = g_{\alpha\beta}(t, x) dx^\alpha dx^\beta = \sum_{\alpha, \beta=0}^n g_{\alpha\beta}(t, x) dx^\alpha dx^\beta.$$

The metric  $\mathbf{g}$  is assumed to be a small asymptotically flat perturbation of the Minkowski metric. More precisely, we suppose  $g_{\alpha\beta}(t, x) \in C^\infty(\mathbb{R}^{1+n})$  and, for some fixed  $\rho > 0$  and  $\delta \ll 1$ ,

$$(H2) \quad \forall \gamma \in \mathbb{N}^{1+n} \quad |\partial_{t,x}^\gamma (g_{\alpha\beta}(t, x) - m_{\alpha\beta})| \leq C_\gamma \delta \langle x \rangle^{-|\gamma|-\rho},$$

with  $(m_{\alpha\beta}) = \text{Diag}(-1, 1, 1, \dots, 1)$  being the standard Minkowski metric and  $\langle x \rangle = \sqrt{1 + |x|^2}$ . An example of such metric can be

$$g_{\alpha\beta} = m_{\alpha\beta} + \delta \langle x \rangle^{-\rho} + \delta \phi(t/\langle x \rangle) \langle x \rangle^{-\rho}$$

with  $\phi \in C_0^\infty$ . Since  $\delta \ll 1$ , it is clear that the metric  $\mathbf{g}$  is a non-trapping perturbation. Let  $g = -\det(g_{\alpha\beta})$ , the Laplace–Beltrami operator associated with  $\mathbf{g}$  is given by

$$\square_{\mathbf{g}} = \sqrt{g}^{-1} \partial_\alpha g^{\alpha\beta} \sqrt{g} \partial_\beta,$$

where  $(g^{\alpha\beta}(t, x))$  denotes the inverse matrix of  $(g_{\alpha\beta}(t, x))$ .

We would also like to investigate the case of radial metric, by which we mean that, when writing out the metric in polar coordinates  $(t, x) = (t, r\omega)$  with  $\omega \in \mathbb{S}^{n-1}$ , we have

$$\mathbf{g} = \tilde{g}_{00}(t, r) dt^2 + 2\tilde{g}_{01}(t, r) dt dr + \tilde{g}_{11}(t, r) dr^2 + \tilde{g}_{22}(t, r) r^2 d\omega^2.$$

In this form, the assumption (H2) on asymptotically flatness is equivalent to the following requirement

$$(H2') \quad |\partial_{t,x}^\gamma (\tilde{g}_{00} + 1, \tilde{g}_{11} - 1, \tilde{g}_{22} - 1, \tilde{g}_{01})| \leq C_\gamma \delta \langle x \rangle^{-|\gamma|-\rho}.$$

Consider the initial value problem for the quasilinear wave equations of the form

$$(5.1) \quad (\square_{\mathbf{g}} u)^I = N^I(u, u), I = 1, 2, \dots, M$$

in which the quadratic nonlinearity  $N = Q + S$  is of the form (1.4). Our construction of solutions will depend on the energy integral method, which requires the quasilinear part to be symmetric (1.5).



In contrast to the null-form system, as in [41], we will be able to avoid the use of the scaling vector field  $S$ , and the vector fields to be used will be labeled as

$$Y = (Y_0, \dots, Y_6) = (\partial, \Omega).$$

For the energy norm, we will use the standard energy norm

$$E_1(u(t)) = \frac{1}{2} \sum_{I=1}^M \int_{\mathbb{R}^n} |\partial u^I(t, x)|^2 dx.$$

The higher order derivatives will be estimated through

$$(5.2) \quad E_m(u(t)) = \sum_{|\alpha| \leq m-1} E_1(Y^\alpha u(t)), \quad m = 2, 3, \dots$$

As for the null-form systems, an important intermediate role will be played by the following local energy norm

$$\mathcal{J}_m(u(t)) = \sum_{I=1}^M \sum_{|\alpha| \leq m-1} \left\| r^{-1/2+\mu} \langle r \rangle^{-\mu'} \left( |\partial \Gamma^\alpha u^I(t)| + \frac{|\Gamma^\alpha u^I(t)|}{r} \right) \right\|_{L^2(\mathbb{R}^n)}^2$$

with  $\mu \in (0, 1/2)$  and  $\mu' > \mu$  to be determined later. This norm is extracted from the local energy norm, which is defined as

$$(5.3) \quad LE_m(t) = \int_0^t \mathcal{J}_m(u(\tau)) d\tau.$$

In the case of  $\mu' = \mu$ , such norm will be the KSS norm and we will denote such norm by  $KSS_m(t)$ . We will choose  $\mu = 1/4$ . When  $n \geq 4$ , the choice for  $\mu'$  will be  $\min(n-2, 2\rho-1, 3)/4$  for the general case and  $\mu' = \min(n-2, 1+2\rho, 3)/4$  for the radial metric. In the case of  $n = 3$ , we will set  $\mu' = \min(2\rho-1, 3)/4$  (and  $\mu' = \min(2\rho+1, 3)/4$  for the radial metric).

In order to describe the solution space, we introduce

$$H_Y^m(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : Y^\alpha f \in L^2, |\alpha| \leq m\},$$

with the norm

$$(5.4) \quad \|f\|_{H_Y^m} = \sum_{|\alpha| \leq m} \|Y^\alpha f\|_{L^2}.$$

Solutions will be constructed in the space  $\dot{H}_Y^m(T)$  obtained by closing the set  $C^\infty([0, T]; C_0^\infty(\mathbb{R}^n))$  in the norm  $\sup_{0 \leq t < T} E_m^{1/2}(u(t))$ . Thus,

$$\dot{H}_Y^m(T) \subset \left\{ u(t, x) : \partial u(t, \cdot) \in \bigcap_{j=0}^{m-1} C^j([0, T]; H_Y^{m-1-j}) \right\}.$$

By Sobolev embedding, it follows that  $\dot{H}_Y^m(T) \subset C^{m-[n+2]/2}([0, T] \times \mathbb{R}^n)$ .

Let us now state our main result precisely.

**Theorem 5.1.** *Let  $n \geq 3$ ,  $\delta \ll 1$ ,  $\rho > 1$  for the general metric and  $\rho > 0$  for the radial metric. Assume that the nonlinear terms in (5.1) satisfy the symmetric condition (1.5). Then there exist constants  $\varepsilon_0, c_0 \ll 1$ , such that the Cauchy problem for (5.1) has a unique global (almost global for  $n = 3$ ) solution  $u \in \dot{H}_Y^\kappa(t)$  for  $t \in [0, T_\varepsilon]$  with*

$$T_\varepsilon = \begin{cases} \infty & n \geq 4 \\ \exp(c_0/\varepsilon) & n = 3 \end{cases},$$

when the initial data satisfy

$$(5.5) \quad E_\kappa^{1/2}(u(0)) = \varepsilon \leq \varepsilon_0, \kappa \geq n + 4.$$

Moreover, the solution satisfies the bounds for some  $C_1 \geq 1$ ,

$$\sup_{t \in [0, T_\varepsilon]} E_\kappa(u(t)) + L E_\kappa(T_\varepsilon) + \delta_{3n} \frac{\varepsilon}{2c_0} K S S_\kappa(T_\varepsilon) \leq C_1^2 \varepsilon^2.$$

**5.1. Commutation with vector fields.** In preparation for the energy estimates, we need to consider the commutation properties of the vector fields  $Y$  with respect to the nonlinear terms.

**Lemma 5.1.** *Let  $u$  be solution of (5.1). Assume that the nonlinearity is of the form (1.4). Then for any  $\alpha \in \mathbb{N}^7$ ,*

$$\square_g Y^\alpha u = \sum_{\beta+\gamma+\mu=\alpha} N_\mu^\alpha(Y^\beta u, Y^\gamma u) + \sum_{|\beta| \leq |\alpha|-1} \left( r_0 \nabla^2 \Gamma^\beta u + r_1 \nabla \Gamma^\beta u \right),$$

in which each  $N_\mu^\alpha$  is a quadratic nonlinearity of the form (1.4), and  $r_m$  with  $m \in \mathbb{N}$  denote functions such that

$$|\partial^\alpha r_m(t, x)| \leq C_\alpha \delta \langle r \rangle^{-\rho-m-|\alpha|} \quad \text{for any } \alpha \in \mathbb{N}^{1+n}.$$

Moreover, if  $|\mu| = 0$ , then  $N_\mu^\alpha = N$ . In addition, if the metric is radial,

$$\square_g Y^\alpha u = \sum_{\beta+\gamma+\mu=\alpha} N_\mu^\alpha(Y^\beta u, Y^\gamma u) + \sum_{|\beta| \leq |\alpha|-1} \left( r_1 \nabla^2 \Gamma^\beta u + r_2 \nabla \Gamma^\beta u \right),$$

*Proof.* It is easy to check that

$$[Y, N](u, v) = YN(u, v) - N(Yu, v) - N(u, Yv)$$

is a quadratic nonlinearity of the form (1.4).

By (H2), we have

$$\square_g = \square + r_0 \partial^2 + r_1 \partial,$$

with  $\square = \text{Diag}(-\partial_t^2 + \Delta, \dots, -\partial_t^2 + \Delta)$ . Recall that

$$[\square, Y_j] = 0,$$

we want to prove the result by induction. It is clear the result is true for  $|\alpha| = 0$ . Now assume that it is true for any  $\alpha$  with  $|\alpha| = m$ . Given  $\alpha_0$  with

$|\alpha_0| = m + 1$ , we can find some  $j$  and  $\alpha$  with  $|\alpha| = m$  and  $Y^{\alpha_0} = Y_j Y^\alpha$ . Then by the inductual assumption, we can calculate as follows

$$\begin{aligned}
\Box_g Y^{\alpha_0} u &= \Box_g Y_j Y^\alpha u \\
&= [\Box_g, Y_j] Y^\alpha u + Y_j \Box_g Y^\alpha u \\
&= [r_0 \partial^2 + r_1 \partial, Y_j] Y^\alpha u \\
&\quad + \sum_{\beta+\gamma+\mu=\alpha} Y_j N_\mu^\alpha (Y^\beta u, Y^\gamma u) + \sum_{|\beta| \leq |\alpha|-1} Y_j \left( r_0 \partial^2 Y^\beta u + r_1 \partial Y^\beta u \right) \\
&= \sum_{|\beta| \leq |\alpha|} \left( r_0 \partial^2 Y^\beta u + r_1 \partial Y^\beta u \right) \\
&\quad + \sum_{\beta+\gamma+\mu=\alpha} \left\{ [Y_j, N_\mu^\alpha] (Y^\beta u, Y^\gamma u) + N_\mu^\alpha (Y_j Y^\beta u, Y^\gamma u) + N_\mu^\alpha (Y^\beta u, Y_j Y^\gamma u) \right\}
\end{aligned}$$

which is of the required form. This completes the proof for the general case. When the metric is radial, the rotational vector fields  $\Omega$  are commutative with  $\Box_g$  and we need only to give the estimate for  $\partial$ . In this case, we have  $[r_{2-j} \partial^j, \partial] = r_{3-j} \partial^j$  with  $j = 1, 2$  and the same argument will give the proof.  $\square$

**5.2. Sobolev-type inequalities.** As in Metcalfe-Sogge [41], we need to use the Sobolev inequalities which does not involve the Lorentz boost operators and scaling vector field, which is also related with the trace estimates.

Recall that we have the following trace estimate (see (1.3) of Fang and Wang [7])

$$r^s \|f(r\omega)\|_{H_\omega^{(n-1)/2-s}} \leq C \|f\|_{\dot{H}^{n/2-s}}, 0 < s < (n-1)/2$$

which gives us (with  $s = 1/4$ )

$$(5.6) \quad r^{1/4} \|f(r\omega)\|_{L_\omega^\infty} \leq C \sum_{|\alpha|+|\beta| \leq (n+3)/2} \|\partial^\alpha \Omega^\beta f\|_{L^2}.$$

Moreover, Sobolev embedding in the polar coordinates gives us for  $r \geq 2$  (see e.g. (2.1) in Metcalfe-Sogge [41])

$$(5.7) \quad r^{(n-1)/2} \|f(r\omega)\|_{L_\omega^\infty} \leq C \sum_{|\alpha|+|\beta| \leq n/2+1} \|\partial^\alpha \Omega^\beta f\|_{L^2(r-1 < |x| < r+1)}.$$

As a direct consequence of the estimates (5.6) and (5.7) is

$$\begin{aligned}
(5.8) \quad & \|fg\|_{L^2}^2 \\
& \leq \|fg\|_{L^2(|x|\leq 2)}^2 + \sum_{j\geq 3} \|fg\|_{L^2(j-1\leq |x|\leq j+1)}^2 \\
& \leq \|r^{1/4}f\|_{L^\infty(|x|\leq 2)}^2 \|r^{-1/4}g\|_{L^2(|x|\leq 2)}^2 \\
& \quad + \sum_{j\geq 3} \|r^{(n-1)/2}f\|_{L^\infty(j-1\leq |x|\leq j+1)}^2 \|r^{-(n-1)/2}g\|_{L^2(j-1\leq |x|\leq j+1)}^2 \\
& \leq C \sum_{|\alpha|\leq (n+3)/2} \|Y^\alpha f\|_{L^2(|x|\leq 3)}^2 \|r^{-1/4}g\|_{L^2(|x|\leq 2)}^2 \\
& \quad + C \sum_{j\geq 3} \sum_{|\alpha|\leq (n+3)/2} \|Y^\alpha f\|_{L^2(j-2\leq |x|\leq j+2)}^2 \|r^{-(n-1)/2}g\|_{L^2(j-1\leq |x|\leq j+1)}^2 \\
& \leq C \|r^{-1/4}\langle r \rangle^{-(n-2)/4}g\|_{L^2}^2 \sum_{|\alpha|\leq (n+3)/2} \|r^{-1/4}\langle r \rangle^{-(n-2)/4}Y^\alpha f\|_{L^2}^2
\end{aligned}$$

**5.3. Energy and local energy estimates.** As in the beginning of Section 3, assume that  $u(t) \in \dot{H}_Y^\kappa(T)$  is a local solution of the initial value problem for (5.1). Our task will be to show that  $E_\kappa(u(t))$  remains finite for all  $t \geq 0$  when  $n \geq 4$  (and for some interval  $[0, \exp(c/\varepsilon)]$  with  $c \ll 1$  when  $n = 3$ ). To do so, we will derive integral inequalities for  $E_\kappa(u(t)) + LE_\kappa(t)$ . If (5.5) holds, then  $E_\kappa^{1/2}(u(0)) = \varepsilon$  and  $E_\kappa(u(t)) + LE_\kappa(T) + (\log(2+T))^{-1}KSS_\kappa(T) < (C_1\varepsilon)^2$  for certain small interval  $t \in [0, T]$ . Define

$$T_0 = \sup\{T : E_\kappa(u(t)) + LE_\kappa(T) + (\log(2+T))^{-1}KSS_\kappa(T) \leq (2C_1\varepsilon)^2, t \in [0, T]\}.$$

Here the constant  $C_1 \geq 1$  will be determined later. All of the following computations will be valid on the time interval  $[0, T_0)$ .

It is easy to check that the same argument of Section 3 gives us the required local energy estimates, that is, by writting

$$\square_{\mathfrak{g}} = \square + (g^{\alpha\beta} - m^{\alpha\beta})\partial_\alpha\partial_\beta + r_1\partial,$$

and applying Lemma 2.4 with  $h^{\alpha\beta} = g^{\alpha\beta} - m^{\alpha\beta}$ , Lemma 5.1 and the symmetry condition (1.5), we have

$$\begin{aligned}
& \sup_{t \in [0, T]} E_m(u(t)) + LE_m(T) + (\log(2+T))^{-1} KSS_m(T) \\
& \leq CE_m(u(0)) + C\delta LE_m(T) \\
& \quad + C \sum_{|\alpha| \leq m-1} \left| \int_0^T C_1^\alpha(t) dt \right| + C \sum_{|\alpha| \leq m-1} \sup_{k \geq 0} \left| \int_0^T C_{2,k}^\alpha(t) dt \right| \\
& \leq CE_m(u(0)) + CC_1(\delta + \varepsilon) LE_m(T) \\
& \quad + C \sum_{|\alpha| = m-1} |E_1^\alpha(T) - E_1^\alpha(0)| + C \sup_{k \geq 0} \sum_{|\alpha| = m-1} |E_{2,k}^\alpha(T) - E_{2,k}^\alpha(0)| \\
& \quad + C \sum_{I=1}^M \sum_{|\alpha| \leq m-1, \beta+\gamma+\mu=\alpha} \sup_{k \geq 0} \left| \int_0^T \int_{\mathbb{R}^3} S_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u)(\partial_t, L_k) \Gamma^\alpha u^I(t) dx dt \right| \\
& \quad + C \sum_{I=1}^M \sum_{|\alpha| \leq m-1, \beta+\gamma+\mu=\alpha, |\gamma| < m-1} \sup_{k \geq 0} \left| \int_0^T \int_{\mathbb{R}^3} Q_\mu^{I,\alpha}(\Gamma^\beta u, \Gamma^\gamma u)(\partial_t, L_k) \Gamma^\alpha u^I(t) dx dt \right| \\
& \quad + C \sum_{I=1}^M \sum_{|\alpha| = m-1} \sup_{k \geq 0} \left| \int_0^T \int_{\mathbb{R}^3} Q_{JK}^{I,\beta\mu\gamma} \partial_\gamma \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K(\partial_t, L_k) \Gamma^\alpha u^I dx dt \right| \\
& \quad + C \sum_{I=1}^M \sum_{|\alpha| = m-1} \sup_{k \geq 0} \left| \int_0^T \int_{\mathbb{R}^3} Q_{JK}^{I,\beta\mu\gamma}(\partial_t, h_k^i \partial_i) \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I dx dt \right|,
\end{aligned}$$

where we have used the assumption

$$(5.9) \quad 2\mu' \leq \rho - 1 + 2\mu, \quad \rho > 1$$

for general metric. When the metric is radial, the assumption can be weakened to be

$$(5.10) \quad 2\mu' \leq \rho + 2\mu, \quad \rho > 0 \text{ (g radial)}.$$

Here, as before,  $L_k = f_k(\partial_r + \frac{1}{r}) = h_k^i(x) \partial_i + h_k(x)$  with  $h_k(x) = f_k(|x|)/|x|$  and  $h_k^i(x) = x^i h_k(x)$ ,  $\mu = 1/4 < \mu' \leq 3/4$ ,

$$(5.11) \quad E_1^\alpha(t) = \sum_{I=1}^M (Q_{JK}^{I,\beta\mu 0} \delta_0^\gamma - \frac{1}{2} Q_{JK}^{I,\beta\mu\gamma}) \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K \partial_\gamma \Gamma^\alpha u^I dx,$$

and

$$(5.12) \quad E_{2,k}^\alpha = \sum_{I=1}^M Q_{JK}^{I,\beta\mu 0} \int_{\mathbb{R}^3} \partial_\beta u^J \partial_\mu \Gamma^\alpha u^K L_k \Gamma^\alpha u^I dx.$$

**5.4. Conclusion of the proof.** We are now ready to complete the proof of Theorem 5.1, by using (5.8) and the local energy estimates in the Subsection 5.3.

We need only to give a better upper bound for

$$\sup_{t \in [0, T]} E_\kappa(u(t)) + LE_\kappa(T) + (\log(2+T))^{-1} KSS_\kappa(T).$$

By the local energy estimates in the Subsection 5.3, we see that there exists an universal constant  $C_2 \geq 1$  such that

$$\begin{aligned} & \sup_{t \in [0, T]} E_\kappa(u(t)) + LE_\kappa(T) + (\log(2+T))^{-1} KSS_\kappa(T) \\ \leq & C_2 E_\kappa(u(0)) + C_2 C_1 (\delta + \varepsilon) LE_\kappa(T) + C_2 \sum_{|\alpha|=\kappa-1} \sup_{k \geq 0} |E_{2,k}^\alpha(T) - E_{2,k}^\alpha(0)| \\ & + C_2 \sum_{|\alpha|=\kappa-1} |E_1^\alpha(T) - E_1^\alpha(0)| \\ & + C_2 \sum_{|\alpha| \leq \kappa-1} \left( \|\partial Y^\alpha u\|_{L_t^\infty L_x^2([0, T] \times \mathbb{R}^n)} + \left\| \frac{1}{r} Y^\alpha u \right\|_{L_t^\infty L_x^2([0, T] \times \mathbb{R}^n)} \right) \\ & \times \sum_{|\alpha| \leq \kappa-1, |\beta| \leq (\kappa-1)/2} \|\partial Y^\beta u \partial Y^\alpha u\|_{L_t^1 L_x^2([0, T] \times \mathbb{R}^n)} \\ \leq & C_2 E_\kappa(u(0)) + C_2 C_1 (\delta + \varepsilon) LE_\kappa(T) + C_2 \sum_{|\alpha|=\kappa-1} \sup_{k \geq 0} |E_{2,k}^\alpha(T) - E_{2,k}^\alpha(0)| \\ & + C_2 \sum_{|\alpha|=\kappa-1} |E_1^\alpha(T) - E_1^\alpha(0)| \\ & + C_2 \sup_{t \in [0, T]} E_\kappa^{1/2}(u(t)) \sum_{|\alpha| \leq \kappa-1, (\kappa+n+2)/2} \|r^{-1/4} \langle r \rangle^{-(n-2)/4} \partial Y^\alpha u\|_{L_t^2 L_x^2([0, T] \times \mathbb{R}^n)}^2, \end{aligned}$$

where in the last inequality, we have used Hardy inequality and (5.8).

Recall the definition (5.11) and (5.12) for  $E_1^\alpha$  and  $E_{2,k}^\alpha$ , we know by Sobolev embedding,

$$\begin{aligned} \sum_{|\alpha|=m-1} (|E_1^\alpha(t)| + \sup_{k \geq 0} |E_{2,k}^\alpha(t)|) & \leq C \|\partial u\|_{L^\infty} E_m(u(t)) \\ & \leq C E_3^{1/2}(u(t)) E_m(u(t)) \\ & \leq C_3 C_1 \varepsilon E_m(u(t)) \end{aligned}$$

for some  $C_3 \geq 2$ .

Based on this observation and the smallness assumption of  $\varepsilon$  and  $\delta$  (such that  $8C_2 C_3 C_1 (\varepsilon + \delta) \leq 1$ ), we can easily obtain the following inequality, when  $(\kappa + n + 2)/2 \leq \kappa - 1$ , i.e.  $\kappa \geq n + 4$ ,

$$\begin{aligned} (5.13) \quad & \sup_{t \in [0, T]} E_\kappa(u(t)) + LE_\kappa(T) + (\log(2+T))^{-1} KSS_\kappa(T) \\ & \leq 2C_2 E_\kappa(u(0)) + 2C_2 E_\kappa^{1/2}(\log(2+T))^{\delta_{3n}} (LE_\kappa(T) + (\log(2+T))^{-1} KSS_\kappa(T)) \end{aligned}$$

on any interval  $[0, T]$  with

$$T < T_0 = \sup\{T : E_\kappa(u(t)) + LE_\kappa(T) + (\log(2+T))^{-1}KSS_\kappa(T) \leq (2C_1\varepsilon)^2, t \in [0, T]\}.$$

If  $n \geq 4$ , we do not have the term  $\log(2+T)$  and

$$(5.14) \quad \sup_{t \in [0, T]} E_\kappa(u(t)) + LE_\kappa(T) + (\log(2+T))^{-1}KSS_\kappa(T) \leq 3C_1^2\varepsilon^2$$

by setting  $C_1 = \sqrt{C_2}$  and  $\delta, \varepsilon \leq \varepsilon_0 = 1/(16C_3C_2C_1)$ . In the case of  $n = 3$ , we have (5.14) for  $T \leq T_\varepsilon$  and  $T < T_0$ , if we set  $C_1 = \sqrt{C_2}$ ,  $\delta, \varepsilon \leq \varepsilon_0 = 1/(16C_3C_2C_1)$ , and

$$T_\varepsilon = \exp\left(\frac{1}{16C_2C_1\varepsilon}\right) - 2 \geq \exp(c_0/\varepsilon)$$

with  $c_0 = 1/(32C_2C_1)$ . When  $n \geq 4$ , we denote  $T_\varepsilon = \infty$ . Here, we may need to let  $\delta$  even smaller such that the assumption in Lemma 2.4 is satisfied.

With this we see the sum of  $E_\kappa^{1/2}(u(t))$  and local energy norms remains less than  $\sqrt{3}C_1\varepsilon$  throughout the interval  $0 \leq t < T_0$  (and  $t < T_\varepsilon$ ). A standard continuity argument shows that  $E_\kappa(u(t))$  is bounded for  $t \in [0, T_\varepsilon)$ , which completes the proof of Theorem 5.1.

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